

MATRIX-VALUED GEGENBAUER POLYNOMIALS

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ABSTRACT. We introduce matrix-valued orthogonal polynomials of arbitrary size, which are analogues of the Gegenbauer or ultraspherical polynomials for the parameter $\nu > 0$. The weight function is given explicitly, and we establish positivity by proving an explicit LDU-decomposition for the weight. Several matrix-valued differential operators of order two and one are shown to be symmetric with respect to the weight, and having the matrix-valued Gegenbauer polynomials as eigenfunctions. Using the parameter ν a simple Rodrigues formula is established. The matrix-valued orthogonal polynomials are connected to the matrix-valued hypergeometric functions, which in turn allows us to give an explicit three-term recurrence relation. We give an explicit non-trivial expression for the matrix entries of the matrix-valued Gegenbauer polynomials in terms of scalar-valued Gegenbauer and Racah polynomials using a diagonalisation procedure for a suitable matrix-valued differential operator. The case $\nu = 1$ reduces to the case of matrix-valued Chebyshev polynomials previously obtained using group theoretic considerations.

1. INTRODUCTION

Matrix-valued orthogonal polynomials have been studied from various different aspects. Originally they occur in the work of M.G. Krein in 1949, who initially studied the corresponding matrix-valued moment problem. Later Krein associated matrix-valued polynomials to differential operators. For more information and applications see the references in [5], [6], [9], [10], [17].

The general theory of matrix-valued orthogonal polynomials has been developed along the lines of orthogonal polynomials, see e.g. [5] and references given there. Explicit examples of matrix-valued orthogonal polynomials have been constructed in several ways, but these matrix-valued orthogonal polynomials are usually limited to low dimensions. The link between representation theory and matrix-valued spherical functions, see [8], gives a way to construct matrix-valued orthogonal polynomials. This approach has been initiated by Grünbaum, Pacharoni, Tirao [9] using invariant differential operators on $SU(3)/U(2)$. Another way to obtain matrix-valued orthogonal polynomials using representation theory can be found in van Pruijssen [20], see also [11], [14], [15]. In particular, all the details for the matrix-valued orthogonal polynomials related to $SU(2) \times SU(2)/SU(2)$ can be found in [14], [15], and these matrix-valued orthogonal polynomials are matrix-valued analogues of the Chebyshev polynomials, which are the (scalar-valued) spherical functions on $SU(2) \times SU(2)/SU(2)$, or, equivalently, the characters on $SU(2)$. The closely related case $SO(4)/SO(3)$ is studied in [18].

Classically the Gegenbauer (or ultraspherical) polynomials $C_n^{(\nu)}(x)$, see [1], [12], [13], for integer values ν can be derived from the Chebyshev polynomials $U_n(x) = C_n^{(1)}(x)$ by repeated

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differentiation. Exploiting this relation, the Rodrigues formula, the second order differential operator, orthogonality relations, three term recurrence relation, etc, for Gegenbauer polynomials can be obtained. We show that we can obtain matrix-valued Gegenbauer polynomials by successive differentiation of the matrix-valued Chebyshev polynomials of [14], [15]. In this way one can obtain the matrix-valued weight function for the matrix-valued Gegenbauer polynomials for integer ν . Using the extension to general ν we find the Ansatz for the general matrix-valued weight function for general matrix-valued Gegenbauer polynomials. This is then the beginning of the study of matrix-valued Gegenbauer polynomials. We start by studying this matrix-valued weight function and its LDU-decomposition in Section 2. The L-matrix in this decomposition has (scalar-valued) Gegenbauer polynomials as its entries. In Section 3 we study symmetric matrix-valued differential operators, which have the matrix-valued Gegenbauer polynomials as eigenfunctions. In Section 3 we also present a Rodrigues formula. The relation between the matrix-valued Gegenbauer polynomials for parameters ν and $\nu + 1$, which are related by differentiation (as lowering operator) and an appropriately defined raising operator, see Section 3, is motivated by the paper [4] by Cantero, Moral and Velázquez. This approach also gives an explicit form for the squared norm matrix for the matrix-valued Gegenbauer polynomials, see Section 3.

Next in Section 4 we establish the connection with Tirao's matrix-valued hypergeometric functions [21]. This gives sufficient additional information in order to establish the three-term recurrence relation explicitly in Section 5. In Section 6 we calculate the matrix entries of the matrix-valued Gegenbauer polynomials in terms of (scalar-valued) Gegenbauer and Racah polynomials. For this we show that we have an explicit matrix-differential operator which is diagonal after conjugation by the L-matrix. The identification with the scalar Gegenbauer polynomial is via a differential operator, whereas the Racah polynomials show up as a solution to a finite three-term recurrence operator. The occurrence of the three-term recurrence operator follows from a set of commuting matrix-valued differential operators and Tirao's [21] theory for matrix-valued hypergeometric functions. We use recent work by Cagliero and Koornwinder [3], which actually gives an explicit inverse of the L-matrix of the LDU-decomposition.

We emphasize that the matrix-valued Gegenbauer polynomials introduced in this paper are given for arbitrary size, and that we give all the basic properties for these matrix-valued Gegenbauer polynomials. On the other hand we know that the scalar-valued Gegenbauer polynomials satisfy many interesting properties, see e.g. [12], [7], so it is natural to expect that more interesting properties for these matrix-valued Gegenbauer polynomials exist. Also, since the case $\nu = 1$ is derived from the group theoretic interpretation, see [14], [15], and also [18], it also to be expected that the matrix-valued Gegenbauer polynomials should have a group theoretic interpretation on the Gelfand pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$, see [20], [11]. For the case of 2×2 -polynomials Pacharoni and Zurrián [19] have been motivated by the representation theory of this Gelfand pair to introduce 2×2 -matrix-valued orthogonal polynomials, and we connect our results to theirs in Remark 2.8(ii).

We have extensively used computer algebra, such as Maple, in order to verify conjectures as well as to come up with explicit conjectures. This means that we have checked many of our results up to a sufficiently large size of the matrices involved. We emphasise that the proofs are all direct, even though it is handy to do some steps in the proofs using computer algebra.

Some more elaborate and relatively uninstrusive proofs are relegated to the appendices. For the reader's convenience the worksheet is available via the first author's webpage.¹

In the development of the matrix-valued Gegenbauer polynomials we need the condition $\nu > 0$, whereas for the scalar-valued Gegenbauer polynomials the condition on the parameter ν is the condition $\nu > -\frac{1}{2}$ for integrability of the weight $(1-x^2)^{\nu-\frac{1}{2}}$ on $(-1, 1)$. In the matrix-valued case the integrability condition remains the same, but we only have a positive definite weight function for $\nu > 0$. So we essentially restrict to the case $\nu > 0$ and comment on the possibility of extending the result to a larger parameter range whenever appropriate.

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2. THE WEIGHT FUNCTION

In this section we define the weight function for the matrix-valued Gegenbauer polynomials explicitly. As stated in Section 1 we have obtained the weight initially for $\nu \in \mathbb{N}$ using the results of [4], and then by continuation in ν . After defining the weight in Definition 2.1, we proceed to state the LDU-decomposition for the weight. This is instrumental in proving that the weight function does indeed satisfy the required properties for a weight function for matrix-valued orthogonal polynomials as in e.g. [10, §2]. We use the standard notation for Gegenbauer (or ultraspherical) polynomials $C_n^{(\nu)}(x)$, see e.g. [1], [12], [13] and Appendix A. The weight function is introduced by defining its matrix entries, which are taken with respect to the standard orthonormal basis $\{e_0, e_1, \dots, e_{2\ell}\}$ of $\mathbb{C}^{2\ell+1}$ for $\ell \in \frac{1}{2}\mathbb{N}$. We suppress ℓ from the notation as much as possible.

Definition 2.1. For $\ell \in \frac{1}{2}\mathbb{N}$ and $\nu > 0$, we define the $(2\ell+1) \times (2\ell+1)$ -matrix-valued functions $W^{(\nu)}$ by

$$\begin{aligned} (W^{(\nu)}(x))_{m,n} &= (1-x^2)^{\nu-1/2} \sum_{t=\max(0, n+m-2\ell)}^m \alpha_t^{(\nu)}(m, n) C_{m+n-2t}^{(\nu)}(x) \\ \alpha_t^{(\nu)}(m, n) &= (-1)^m \frac{n! m! (m+n-2t)!}{t! (2\nu)_{m+n-2t} (\nu)_{n+m-t}} \frac{(\nu)_{n-t} (\nu)_{m-t}}{(n-t)! (m-t)!} \frac{(n+m-2t+\nu)}{(n+m-t+\nu)} \\ &\quad \times (2\ell-m)! (n-2\ell)_{m-t} (-2\ell-\nu)_t \frac{(2\ell+\nu)}{(2\ell)!} \end{aligned} \quad (2.1)$$

where $n, m \in \{0, 1, \dots, 2\ell\}$ and $n \geq m$. The matrix is extended to a symmetric matrix, $(W^{(\nu)}(x))_{m,n} = (W^{(\nu)}(x))_{n,m}$. Finally, put $W^{(\nu)}(x) = (1-x^2)^{\nu-1/2} W_{\text{pol}}^{(\nu)}(x)$.

¹<http://www.math.ru.nl/~koelink>

Remark 2.2. (i) By construction $W^{(\nu)}: (-1, 1) \rightarrow M_{2\ell+1}(\mathbb{C})$, the space of $(2\ell+1) \times (2\ell+1)$ -matrices, and each of the matrix entries times a monomial x^k is absolutely integrable on $[-1, 1]$ for $k \in \mathbb{N}$. Only requiring integrability and finite moments shows that we can extend Definition 2.1 to $\nu > -\frac{1}{2}$, where the case $\nu = 0$ is defined using the limit $\lim_{\nu \rightarrow 0} \frac{n+2\nu}{2\nu} C_n^{(\nu)}(x) = T_n(x)$, the Chebyshev polynomials of the first kind. Note that for $\nu = 0$ the sum over t in each matrix entry reduces to a single term; $\left(W_{\text{pol}}^{(0)}(x)\right)_{m,n} = (2\ell) T_{n-m}(x)$ for $n \geq m$, $\ell \geq \frac{1}{2}$, so that $W^{(0)}(x)$ is a persymmetric Toeplitz matrix. However, $\det\left(W_{\text{pol}}^{(0)}(x)\right) = 0$ for $\ell \geq 1$. To see this note that $(1, -2x, 1, 0, \dots, 0)^t$ is in the kernel of $W_{\text{pol}}^{(0)}(x)$ for any x , using the three-term recurrence $2xT_n(x) = T_{n+1}(x) + T_{n-1}(x)$ and $T_0(x) = 1$, $T_1(x) = x$. Similarly, $(0, \dots, 0, 1, -2x, 1, 0, \dots, 0)^t$ is in the kernel as well, so that the kernel has at least codimension 2.

(ii) The case $\nu = 1$ gives back the weight function as in [14, Thm. 5.4], [15, (1.1)].

(iii) In case $\ell = 0$ we have $\left(W^{(\nu)}(x)\right)_{m,n} = \nu(1-x^2)^{\nu-1/2}$, a multiple of the (scalar) weight for the Gegenbauer polynomials, see [1], [12], [13], Appendix A. In this paper we consider $\ell \geq \frac{1}{2}$.

(iv) Note that $\left(W_{\text{pol}}^{(\nu)}(x)\right)_{n,m}$ is a polynomial of degree $n+m$ in case $n+m \leq 2\ell$ and of degree $n+m-2\ell$ in case $n+m \geq 2\ell$. Moreover, the summation over t can run over 0 to m , since $(n-2\ell)_{m-t} = 0$ for $n+m-2\ell-t > 0$.

Theorem 2.3. For $\nu > 0$, $W^{(\nu)}(x)$ has the following LDU-decomposition

$$W^{(\nu)}(x) = (1-x^2)^{\nu-1/2} L^{(\nu)}(x) T^{(\nu)}(x) L^{(\nu)}(x)^t, \quad x \in (-1, 1), \quad (2.2)$$

where $L^{(\nu)}: [-1, 1] \rightarrow M_{2\ell+1}(\mathbb{C})$ is the unipotent lower triangular matrix-valued polynomial

$$\left(L^{(\nu)}(x)\right)_{m,k} = \begin{cases} 0 & \text{if } m < k \\ \frac{m!}{(m-k)! k!} {}_2F_1\left(\begin{matrix} -m+k, m+k+2\nu \\ \frac{1}{2}+k+\nu \end{matrix}; \frac{(1-x)}{2}\right) & \text{if } m \geq k. \end{cases}$$

and $T^{(\nu)}: [-1, 1] \rightarrow M_{2\ell+1}(\mathbb{C})$ is the diagonal matrix-valued polynomial

$$\begin{aligned} \left(T^{(\nu)}(x)\right)_{k,k} &= t_k^{(\nu)} (1-x^2)^k, \\ t_k^{(\nu)} &= \frac{k! (\nu)_k}{(\nu+1/2)_k} \frac{(2\nu+2\ell)_k (2\ell+\nu)}{(2\ell-k+1)_k (2\nu+k-1)_k}. \end{aligned}$$

The proof of Theorem 2.3 is based on an integral evaluation and manipulations of hypergeometric series, and is relegated to Appendix A. From the proof given we can immediately conclude that Theorem 2.3 can be extended to $\nu > -\frac{1}{2}$, $\nu \neq 0$. Note that we can write the matrix entries of $L^{(\nu)}$ in terms of Gegenbauer polynomials; for $m \geq k$

$$\left(L^{(\nu)}(x)\right)_{m,k} = \frac{m!}{k! (2\nu+2k)_{m-k}} C_{m-k}^{(\nu+k)}(x) \quad (2.3)$$

The matrix $L(x)$ is invertible, and it is remarkable that the inverse is again completely described in terms of hypergeometric series. Explicitly, see Cagliero and Koornwinder [3, (4.7)],

Thm. 4.1],

$$\left((L^{(\nu)}(x))^{-1} \right)_{k,n} = \frac{k!}{n! (2\nu + k + n - 1)_{k-n}} C_{k-n}^{(1-\nu-k)}(x), \quad k \geq n, \quad (2.4)$$

where we follow the convention of [3, §§2, 3] for $C_n^{(-p)}(x)$ for $p \in \mathbb{N}$.

We consider some consequences of Theorem 2.3. First we observe that

$$W_{\text{pol}}^{(\nu)}(1)_{i,j} = (2\ell + \nu), \quad W_{\text{pol}}^{(\nu)}(-1)_{i,j} = (-1)^{i-j} (2\ell + \nu). \quad (2.5)$$

This follows from the observation $L^{(\nu)}(1)_{m,k} = \binom{m}{k}$ (taken as 0 in case $m < k$) and $T_{m,k}^{(\nu)}(\pm 1) = \delta_{m,k} \delta_{k,0} t_0^{(\nu)}$, and for $x = -1$ we use that the Gegenbauer polynomials are symmetric to reduce to the case $x = 1$.

Note secondly that $t_0^{(\nu)} = (2\ell + \nu)$, $t_1^{(\nu)} = \frac{(\nu+\ell)(\nu+2\ell)}{(\nu+1/2)(2\ell)}$, which are both positive for $\nu > -\frac{1}{2}$ and $\ell \geq \frac{1}{2}$. However, for $k \geq 2$ and $\ell > \frac{1}{2}$ we see that $t_k^{(\nu)}$ equals ν times a positive factor.

Corollary 2.4. *For $\nu > 0$ we have $\det(W^{(\nu)}(x)) = (1 - x^2)^{(2\ell+1)(\ell+\nu-1/2)} \prod_{k=0}^{2\ell} t_k^{(\nu)}(\ell)$.*

Corollary 2.4 extends [15, Cor. 2.3], which in turns was conjectured in [14, Conj. 5.8]. Note that Corollary 2.4 remains valid for $\nu > -\frac{1}{2}$, $\nu \neq 0$, since Theorem 2.3 is valid there as well. For $\nu = 0$, $\ell \geq 1$ taking into account Remark 2.2(i), Corollary 2.4 remains valid as well, since $t_k^{(0)} = 0$ for $k \geq 2$, $\ell \geq 1$.

Corollary 2.5. *$W^{(\nu)}(x)$ is strictly positive definite for $x \in (-1, 1)$ for $\nu > 0$. In case $-\frac{1}{2} < \nu \leq 0$ and $\ell = \frac{1}{2}$ we have that $W^{(\nu)}(x)$ is strictly positive definite. In case $-\frac{1}{2} < \nu < 0$ and $\ell > \frac{1}{2}$, $W^{(\nu)}(x)$ has signature $(2, 2\ell - 1)$. In case $\nu = 0$ and $\ell > \frac{1}{2}$, $W^{(0)}(x)$ is indefinite.*

In case $\nu > 0$, we see that $W^{(\nu)}(x)$ is a strictly positive definite matrix for $x \in (-1, 1)$, so that the results of Grünbaum and Tirao [10, §2] apply. So from now on we restrict to the case $\nu > 0$. In particular, there exists a family of matrix-valued orthogonal polynomials. We denote by $P_n^{(\nu)}$ the corresponding monic matrix-valued orthogonal polynomial of degree n ;

$$\int_{-1}^1 P_n^{(\nu)}(x) W^{(\nu)}(x) (P_m^{(\nu)}(x))^* dx = \delta_{n,m} H_n^{(\nu)}, \quad (2.6)$$

$$P_n^{(\nu)}(x) = x^n \text{Id} + x^{n-1} P_{n,n-1}^{(\nu)} + \cdots + x P_{n,1}^{(\nu)} + P_{n,0}^{(\nu)}, \quad P_{n,n}^{(\nu)} = \text{Id}, P_{n,i}^{(\nu)} \in M_{2\ell+1}(\mathbb{C}),$$

where $H_n^{(\nu)}$ is a strictly positive definite matrix. We denote the matrix-valued inner product by

$$\langle P, Q \rangle^{(\nu)} = \int_{-1}^1 P(x) W^{(\nu)}(x) (Q(x))^* dx, \quad \nu > 0. \quad (2.7)$$

For the case $\nu = 1$, these matrix-valued polynomials have been derived from a group theoretic setting [14], and further studied in [15], see also [18]. For details on the general theory of matrix-valued orthogonal polynomials we refer to [5], [10], [17]. In particular, the integration in (2.6), (2.7) is per matrix-entry. In particular, the case $n = 0$ in (2.6) can be

evaluated explicitly using the orthogonality relations (A.1) for the Gegenbauer polynomials, to see that $H_0^{(\nu)}$ is a diagonal matrix with entries

$$(H_0^{(\nu)})_{k,k} = \alpha_k^{(\nu)}(k, k) \int_{-1}^1 (1-x^2)^{\nu-1/2} dx = (2\ell + \nu) \sqrt{\pi} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + 1)} \frac{k! (2\ell - k)! (\nu + 1)_{2\ell}}{(2\ell)! (\nu + 1)_k (\nu + 1)_{2\ell-k}} \quad (2.8)$$

Before we study the polynomials closely, we determine the commutant of the weight.

Proposition 2.6. *Let $\nu > 0$. The commutant algebra $A^{(\nu)} = \{T \in M_{2\ell+1}(\mathbb{C}) \mid [T, W^{(\nu)}(x)] = 0 \forall x \in (-1, 1)\}$ is generated by J , where $J \in M_{2\ell+1}(\mathbb{C})$ is the involution defined by $e_j \mapsto e_{2\ell-j}$.*

Proof. Observe that $W^{(\nu)}(x)J = JW^{(\nu)}(x)$ for all $x \in (-1, 1)$ if and only if $(W^{(\nu)}(x))_{2\ell-n,m} = (W^{(\nu)}(x))_{n,2\ell-m}$ for all n, m and all x . This means that the weight matrix $W^{(\nu)}(x)$ is persymmetric (or orthosymmetric). Using Definition 2.1 and comparing coefficients of the Gegenbauer polynomials we need to prove

$$\alpha_t^{(\nu)}(\min(2\ell - m, n), \max(2\ell - m, n)) = \alpha_{t+m-n}^{(\nu)}(\min(2\ell - n, m), \max(2\ell - n, m)).$$

This is straightforwardly verified from the expression in Definition 2.1. Note also that then the summation ranges in Definition 2.1 match.

Conversely, if $T \in A^{(\nu)}$ we see that T has to commute with $W_{\text{pol}}^{(\nu)}(x) = \sum_{k=0}^{2\ell} W_k C_k^{(\nu)}(x)$ for symmetric $W_k \in M_{2\ell+1}(\mathbb{C})$. In particular, $[T, W_k] = 0$ for all k . We want to prove that this implies that T is a linear combination of the identity and J as in the proof of [14, Prop. 5.5].

In order to use the proof of [14, Prop. 5.5] we note that the role of C in the proof of [14, Prop. 5.5] is now played by $W_{2\ell}$ and the role of C' in [14, Prop. 5.5] by $W_{2\ell-1}$. Now $(W_{2\ell})_{m,n}$ is non-zero if and only if $n + m = 2\ell$, and $(W_{2\ell})_{m,2\ell-m} = \frac{2\ell}{(2\nu)_{2\ell}(\nu)_{2\ell}} (2\ell - m)! m! (\nu)_m (\nu)_{2\ell-m}$. Since $\nu > 0$ we have $(W_{2\ell})_{n,2\ell-n} = (W_{2\ell})_{m,2\ell-m}$ if and only if $n = 2\ell - m$ or $n = m$, so that $W_{2\ell}$ has the required properties as the matrix C in the proof of [14, Prop. 5.5]. To see that $(W_{2\ell})_{n,2\ell-n} = (W_{2\ell})_{m,2\ell-m}$ if and only if $n = 2\ell - m$ or $n = m$ assume $0 \leq n < m \leq \ell$ and $(2\ell - m)! m! (\nu)_m (\nu)_{2\ell-m} = (2\ell - n)! n! (\nu)_n (\nu)_{2\ell-n}$. Taking out the common term $n! (2\ell - m)! (\nu)_n (\nu)_{2\ell-m}$, which is zero for unallowed values of ν , namely $\{0, -1, \dots, -\max(n-1, 2\ell - m - 1)\}$, we find $(2\ell - m + 1)_{m-n} (\nu + 2\ell - m)_{m-n} = (n + 1)_{m-n} (\nu + n)_{m-n}$. Because of the conditions on ν , n and m we obtain $(n + 1)_{m-n} < (2\ell - m + 1)_{m-n}$ and its extension $(\nu + n)_{m-n} < (\nu + 2\ell - m)_{m-n}$. So this proves the statement.

Moreover, $(W_{2\ell-1})_{n,m}$ is nonzero if and only if $|n + m - 2\ell| = 1$. In this case, we also have $W_{2\ell-1}$ is symmetric and persymmetric, and apart from these symmetries all entries of $W_{2\ell-1}$ are different. These are precisely the required properties to make the proof of [14, Prop. 5.5] work. In order to see this note that $(W_{2\ell})_{m+1,2\ell-m} = \frac{2\ell+\nu}{(2\ell)(2\nu)_{2\ell-1}(\nu)_{2\ell-1}} (2\ell - m)! (m + 1)! (\nu)_m (\nu)_{2\ell-1-m}$. Assuming $0 \leq n < m \leq \ell - \frac{1}{2}$ and $(2\ell - n)! (n + 1)! (\nu)_n (\nu)_{2\ell-1-n} = (2\ell - m)! (m + 1)! (\nu)_m (\nu)_{2\ell-1-m}$, we derive a contradiction in the same way.

Now the proof can be completed analogously to [14, Prop. 5.5]. \square

The proof extends again to $\nu > -\frac{1}{2}$, $\nu \neq 0$, by a slightly more careful analysis. It breaks down in case $\nu = 0$, since the non-zero matrix entries of the corresponding $W_{2\ell}$ and $W_{2\ell-1}$ are all the same.

Remark 2.7. It follows that $A^{(\nu)}$ is a two-dimensional algebra, and that the invariant subspaces of $\mathbb{C}^{2\ell+1}$ for $W^{(\nu)}(x)$ are the ± 1 -eigenspaces of J . Then the restrictions of $W^{(\nu)}(x)$ to the ± 1 -eigenspaces are irreducible. Explicitly,

$$W^{(\nu)}(x) = Y_\ell^t \begin{pmatrix} W_+^{(\nu)}(x) & 0 \\ 0 & W_-^{(\nu)}(x) \end{pmatrix} Y_\ell, \quad Y_\ell \in \text{SO}(2\ell+1), \quad (2.9)$$

$$Y_{p+\frac{1}{2}} = \frac{1}{2}\sqrt{2} \begin{pmatrix} I_{p+1} & J_{p+1} \\ -J_{p+1} & I_{p+1} \end{pmatrix}, \quad Y_p = \frac{1}{2}\sqrt{2} \begin{pmatrix} I_p & 0 & J_p \\ 0 & \sqrt{2} & 0 \\ -J_p & 0 & I_p \end{pmatrix}, \quad p \in \mathbb{N},$$

where I_p denotes the identity as $p \times p$ -matrix and J_p the antidiagonal matrix (with all 1's) in $M_p(\mathbb{C})$, see also [14, §6]. Note that $W_+^{(\nu)}(x) \in M_{\lceil \ell+\frac{1}{2} \rceil}(\mathbb{C})$ and $W_-^{(\nu)}(x) \in M_{\lfloor \ell+\frac{1}{2} \rfloor}(\mathbb{C})$. The corresponding monic orthogonal polynomials are denoted $P_{\pm, n}^{(\nu)}$.

The involution $F \in M_{2\ell+1}(\mathbb{C})$, $e_j \mapsto (-1)^j e_j$ satisfies $W^{(\nu)}(x)F = FW^{(\nu)}(-x)$, since $C_n^{(\nu)}(-x) = (-1)^n C_n^{(\nu)}(x)$. Since $FJ = (-1)^{2\ell} JF$, we find in case the dimension $2\ell+1$ is even that the weights $W_+^{(\nu)}(-x) = FJW_-^{(\nu)}(x)JF$ (where F and J are now the corresponding diagonal and antidiagonal operators in $M_{(2\ell+1)/2}(\mathbb{C})$.) For odd $2\ell+1$ there is no link between the weights $W_\pm^{(\nu)}$. So this is analogous to the case $\nu = 1$ of [14, §6].

Remark 2.8. (i) For $\ell = \frac{1}{2}$ we have

$$W^{(\nu)}(x) = (1+\nu)Y_\ell^t \begin{pmatrix} (1-x)^{\nu-\frac{1}{2}}(1+x)^{\nu+\frac{1}{2}} & 0 \\ 0 & (1-x)^{\nu+\frac{1}{2}}(1+x)^{\nu-\frac{1}{2}} \end{pmatrix} Y_\ell,$$

so in case $\ell = \frac{1}{2}$ we find a combination of two scalar Jacobi weights with parameters (α, β) equal to $(\nu + \frac{1}{2}, \nu - \frac{1}{2})$ and $(\nu - \frac{1}{2}, \nu + \frac{1}{2})$. So the polynomials $P_n^{(\nu)}(x)$ can also be written in terms of Jacobi polynomials, and the relation $W_-^{(\nu)}(-x) = W_+^{(\nu)}(-x)$ is immediate.

(ii) Pacharoni and Zurrián [19, §§1, 4] introduce a family of 2×2 weight functions by

$$W_{p,n}(x) = (1-x^2)^{n/2-1} \begin{pmatrix} px^2 + n - p & -nx \\ -nx & (n-p)x^2 + p \end{pmatrix}, \quad 0 < p < n$$

with $p, n \in \mathbb{N}$. The weight $W_{p,n}(x)$ is reducible only for $p = \frac{1}{2}n$ and satisfies a symmetry $p \leftrightarrow n - p$, see [19, §4]. In our set-up we have 2×2 irreducible constituents in the weight function for $\ell \in \{1, \frac{3}{2}, 2\}$. Explicitly, the 2×2 irreducible weights are

$$W_+^{(\nu)}(x) = (1-x^2)^{\nu-\frac{1}{2}} \frac{2(2+\nu)}{1+2\nu} \begin{pmatrix} (\nu+1)x^2 + \nu & \frac{1+2\nu}{\sqrt{2}}x \\ \frac{1+2\nu}{\sqrt{2}}x & \frac{1}{2}(\nu x^2 + \nu + 1) \end{pmatrix}$$

$$W_+^{(\nu)}(x) = \frac{(3+\nu)(1-x^2)^{\nu-\frac{1}{2}}(1+x)}{1+2\nu} \begin{pmatrix} (4+2\nu)x(x-1) + 1 + 2\nu & 2(\nu+1)x - 1 \\ 2(\nu+1)x - 1 & \frac{1}{3}(2\nu x(x+1) + 3 + 2\nu) \end{pmatrix}$$

$$W_-^{(\nu)}(x) = \frac{(\nu+4)(\nu+2)}{(2\nu+1)(2\nu+3)} (1-x^2)^{\nu+1/2} \begin{pmatrix} \nu x^2 + \nu + 3 & 2(2\nu+3)x \\ 2(2\nu+3)x & 4((\nu+3)x^2 + \nu) \end{pmatrix}.$$

occurring in respectively the cases $\ell = 1$, $\ell = \frac{3}{2}$, $\ell = 2$. The explicit weights of size 2×2 for $\ell = 1$ and $\ell = 2$ can be matched to $DW_{p,n}(x)D^t$ for a suitable diagonal matrix D with

$(n, p) = (2\nu + 1, \nu + 1)$ and $(n, p) = (2\nu + 3, \nu + 3)$ respectively. The submatrices for $\ell = \frac{3}{2}$ do not correspond to any $W_{n,p}$ of [19] as follows from the explicit expression for $W_+^{(\nu)}(x)$, and then the same statement also follows for $W_-^{(\nu)}(x)$ by Remark 2.7.

3. POLYNOMIAL EIGENFUNCTIONS OF MATRIX-VALUED DIFFERENTIAL OPERATORS

In this section we establish a first order and a second order matrix-valued differential equation which are symmetric with respect to the weight function. In particular, see e.g. [10], this implies that the monic matrix-valued orthogonal polynomials are eigenfunctions for these differential operators. The differential operators have been obtained initially for integer ν as explained in Section 1, and we supply a direct proof of their symmetry for general $\nu > 0$. The freedom in ν is used to factorize a second-order matrix-valued differential operator, and to make a link to Cantero, Moral, Velázquez [4]. This leads to raising and lowering operators which can be used to give an explicit evaluation of the squared norm and to establish a Rodrigues formula.

Theorem 3.1. *For $\nu > 0$, let $D^{(\nu)}$ and $E^{(\nu)}$ be the matrix-valued differential operators*

$$D^{(\nu)} = (1 - x^2) \frac{d^2}{dx^2} + \left(\frac{d}{dx} \right) (C^{(\nu)} - xU^{(\nu)}) - V^{(\nu)}, \quad E^{(\nu)} = \left(\frac{d}{dx} \right) (xB_1^{(\nu)} + B_0^{(\nu)}) + A_0^{(\nu)},$$

where the matrices $C^{(\nu)}$, $U^{(\nu)}$, $V^{(\nu)}$, $B_0^{(\nu)}$, $B_1^{(\nu)}$ and $A_0^{(\nu)}$ are given by

$$\begin{aligned} C^{(\nu)} &= \sum_{i=0}^{2\ell} (2\ell - i) E_{i,i+1} + \sum_{i=0}^{2\ell} i E_{i,i-1}, & U^{(\nu)} &= (2\ell + 2\nu + 1)I, \\ V^{(\nu)} &= - \sum_{i=0}^{2\ell} i(2\ell - i) E_{i,i} + (\nu - 1)(2\ell + \nu + 1)I, \\ B_0^{(\nu)} &= \sum_{i=0}^{2\ell} \frac{(2\ell - i)}{2\ell} E_{i,i+1} - \sum_{i=0}^{2\ell} \frac{i}{2\ell} E_{i,i-1}, & B_1^{(\nu)} &= - \sum_{i=0}^{2\ell} \frac{(\ell - i)}{\ell} E_{i,i}, \\ A_0^{(\nu)} &= \sum_{i=0}^{2\ell} \left(\frac{(2\ell + 2)(i - 2\ell)}{2\ell} - (\nu - 1) \frac{(\ell - i)}{\ell} \right) E_{i,i}. \end{aligned}$$

Then $D^{(\nu)}$ and $E^{(\nu)}$ are symmetric with respect to the weight $W^{(\nu)}$, and $D^{(\nu)}$ and $E^{(\nu)}$ commute. Moreover for every integer $n \geq 0$, the monic matrix-valued Gegenbauer polynomials, see (2.6), are eigenfunctions;

$$\begin{aligned} P_n^{(\nu)} D^{(\nu)} &= \Lambda_n(D^{(\nu)}) P_n^{(\nu)}, & \Lambda_n(D^{(\nu)}) &= \sum_{i=0}^{2\ell} (i(2\ell - i) - (n + \nu - 1)(2\ell + \nu + n + 1)) E_{i,i}, \\ P_n^{(\nu)} E^{(\nu)} &= \Lambda_n(E^{(\nu)}) P_n^{(\nu)}, & \Lambda_n(E^{(\nu)}) &= \sum_{i=0}^{2\ell} \frac{(\ell + 1)(i - 2\ell) - n(\ell - i) - (\nu - 1)(\ell - i)}{\ell} E_{i,i}. \end{aligned}$$

Note that the notation implies that matrix-valued functions act from the right after taking derivatives, so e.g. for a matrix-valued polynomial P we have

$$(PE^{(\nu)})(x) = \left(\frac{dP}{dx}(x) \right) (xB_1^{(\nu)} + B_0^{(\nu)}) + A_0^{(\nu)},$$

where the derivative $\frac{dP}{dx}(x)$ is obtained by taking derivatives entry-wise. In Theorem 3.1 we use the superscript ν even if some of the matrix-valued polynomials occurring as coefficients are actually independent of ν , in order to emphasise the ν -dependence in the differential operator.

Theorem 2.3 for $\nu = 1$ corresponds to [15, Thm. 3.1]. Initially we have obtained the operators of Theorem 3.1 for the case $\nu \in \mathbb{N} \setminus \{0\}$ by repeated differentiation from the case $\nu = 1$. Next using continuation with respect to the parameter we obtain the explicit expressions for arbitrary $\nu > 0$. However, for general $\nu > 0$ we need to prove Theorem 3.1 directly.

Remark 3.2. Consider the generator J of the commutant algebra, see Proposition 2.6, then a direct calculation shows that $JD^{(\nu)}J = D^{(\nu)}$. So, by Remark 2.7, we see that $D^{(\nu)}$ restricts to a second order matrix-valued differential operator on the space of polynomials with values in the $+1$, respectively -1 , eigenspace of J . In particular, the restrictions $D_{\pm}^{(\nu)}$ have $P_{\pm,n}^{(\nu)}$ as eigenfunctions. More interestingly, $J(E^{(\nu)} + (\ell + 1)I)J = -(E^{(\nu)} + (\ell + 1)I)$, cf. [15, §7] for a group theoretic interpretation in case $\nu = 1$. We have refrained from redefining $E^{(\nu)}$ in order to comply with the notation of [15] for the case $\nu = 1$. Note that $E^{(\nu)} + (\ell + 1)I$ maps polynomials taking values in the $+1$ eigenspace of J to polynomials taking values in the -1 eigenspace of J and vice versa. Note that $E^{(\nu)}$ is not defined in the excluded case $\ell = 0$, and that we only have a first-order differential equation with genuine matrix-valued orthogonal polynomials as eigenfunctions. However, in the reducible case $\ell = \frac{1}{2}$, see Remark 2.8(i), we obtain from the eigenvalue equation for $E^{(\nu)}$ the identity, cf. [14, §8.2.1]

$$(1+x) \frac{dP_n^{(\nu-1/2, \nu+1/2)}}{dx}(x) + \left(\nu + \frac{1}{2}\right) P_n^{(\nu-1/2, \nu+1/2)}(x) = \left(n + \nu + \frac{1}{2}\right) P_n^{(\nu+1/2, \nu-1/2)}(x)$$

as well as the one obtained by switching $x \leftrightarrow -x$, cf. Remark 2.7, where $P_n^{(\alpha, \beta)}(x)$ is the Jacobi polynomial normalized in the standard way [12], [13].

Proof. According to [6, Thm 3.1] (with $A_2 = 0$) the symmetry conditions for $E^{(\nu)}$ are

$$W^{(\nu)}(x)(xB_1^{(\nu)} + B_0^{(\nu)})^* + (xB_1^{(\nu)} + B_0^{(\nu)})W^{(\nu)}(x) = 0, \quad (3.1)$$

$$-\frac{d}{dx}[(xB_1^{(\nu)} + B_0^{(\nu)})W^{(\nu)}(x)] + A_0^{(\nu)}W^{(\nu)}(x) = W^{(\nu)}(x)(A_0^{(\nu)})^*. \quad (3.2)$$

Similarly, [6, Thm 3.1] (with $A_2 = (1 - x^2)\text{Id}$) show that the symmetry conditions for $D^{(\nu)}$ are

$$\begin{aligned} 2\frac{d}{dx}((1-x^2)W^{(\nu)}(x)) - (C^{(\nu)} - xU^{(\nu)})W^{(\nu)}(x) &= W^{(\nu)}(x)(C^{(\nu)} - xU^{(\nu)})^* \\ \frac{d^2}{dx^2}((1-x^2)W^{(\nu)}(x)) - \frac{d}{dx}((C^{(\nu)} - xU^{(\nu)})W^{(\nu)}(x)) - V^{(\nu)}W^{(\nu)}(x) &= W^{(\nu)}(x)(V^{(\nu)})^*. \end{aligned} \quad (3.3)$$

Differentiating the first equation and using the result in the second of (3.3), and writing $W^{(\nu)}(x) = (1 - x^2)^{\nu-1/2}W_{\text{pol}}^{(\nu)}(x)$ and canceling common terms shows that it suffices to prove

$$(4\nu + 2 + (C^{(\nu)} - xU^{(\nu)}))W_{\text{pol}}^{(\nu)}(x) - 2(1 - x^2)W_{\text{pol}}^{(\nu)}(x)' + W_{\text{pol}}^{(\nu)}(x)(C^{(\nu)} - xU^{(\nu)})^* = 0, \quad (3.4)$$

$$(W^{(\nu)}(x)(C^{(\nu)} - xU^{(\nu)})^* - (C^{(\nu)} - xU^{(\nu)})W^{(\nu)}(x))' = 2(W^{(\nu)}(x)V^{(\nu)} - V^{(\nu)}W^{(\nu)}(x)) \quad (3.5)$$

The proofs of (3.1), (3.2), (3.4), (3.5) rest on its verification using the explicit formulas for $W^{(\nu)}$ of Definition 2.1 and properties of Gegenbauer polynomials. This verification only involves the scalar Gegenbauer polynomials, and the proof is relegated to Appendix B.

We need to check the boundary conditions for $D^{(\nu)}$, see [6, Thm 3.1],

$$\lim_{x \rightarrow \pm 1} (1 - x^2)W^{(\nu)}(x) = 0, \quad \lim_{x \rightarrow \pm 1} ((1 - x^2)W^{(\nu)}(x))' - (C^{(\nu)} - xU^{(\nu)})W^{(\nu)}(x) = 0$$

of which the first follows for all $\nu > -\frac{1}{2}$, so in particular for $\nu > 0$. To prove the second limit we use the first equation of (3.3), to rewrite this as $\lim_{x \rightarrow \pm 1} W^{(\nu)}(x)(C^{(\nu)} - xU^{(\nu)})^* - (C^{(\nu)} - xU^{(\nu)})W^{(\nu)}(x) = 0$. Since $W^{(\nu)}(x) = (1 - x^2)^{\nu-1/2}W_{\text{pol}}^{(\nu)}(x)$ it suffices to show that the matrix-valued polynomial $(C^{(\nu)} - xU^{(\nu)})W_{\text{pol}}^{(\nu)}(x)$ and $W_{\text{pol}}^{(\nu)}(x)(C^{(\nu)} - xU^{(\nu)})^*$ are equal for $x = \pm 1$. Since $U^{(\nu)}$ is a real multiple of the identity, it suffices to check that $C^{(\nu)}W_{\text{pol}}^{(\nu)}(x)$ and $W_{\text{pol}}^{(\nu)}(x)(C^{(\nu)})^*$ agree for $x = \pm 1$. Using (2.5) and the explicit expression for $C^{(\nu)}$ as a tridiagonal matrix in Theorem 3.1 this is a simple check.

For the first order operator $E^{(\nu)}$, the boundary condition is $\lim_{x \rightarrow \pm 1} (xB_1^{(\nu)} + B_0^{(\nu)})(1 - x^2)^{\nu-1/2}W_{\text{pol}}^{(\nu)}(x) = 0$, which corresponds to [6, Thm 3.1]. Since $(xB_1^{(\nu)} + B_0^{(\nu)})W_{\text{pol}}^{(\nu)}(x) = 0$ for $x = \pm 1$ by (2.5) and the explicit expression for $B_1^{(\nu)}$ and $B_0^{(\nu)}$, the boundary condition follows for all $\nu > 0$.

It then follows from [10] that the monic orthogonal polynomials $P_n^{(\nu)}$ introduced in (2.6) are eigenfunctions for the symmetric operators $E^{(\nu)}$, $D^{(\nu)}$. The eigenvalue matrix, which is acting from the left, follows by comparing leading coefficients. Since the eigenvalue matrices for $E^{(\nu)}$, $D^{(\nu)}$ are diagonal, they commute. By [10] this implies that $E^{(\nu)}$ and $D^{(\nu)}$ commute. \square

Classically, the Gegenbauer polynomials with parameter $\nu + 1$ are obtained from the Gegenbauer polynomials with parameter ν by differentiation. For matrix-valued orthogonal polynomials Cantero, Moral, Velázquez [4] established criteria for the derived matrix-valued orthogonal polynomials to be again matrix-valued orthogonal polynomials. We adapt the notation to their result [4, Thm. 3.14]. In order to establish the connection with [4], we consider the following second order differential operator for $\nu > 0$, for which the notation is motivated by (3.7),

$$D_{(\Phi, \Psi)}^{(\nu)} = (E^{(\nu)})^2 + (2\ell + 2)E^{(\nu)} + \left(\frac{(\ell + \nu)^2}{\ell^2} \right) D^{(\nu)} + \frac{\nu(\nu - 1)(2\ell + \nu + 1)(\nu + 2\ell)}{\ell^2} \text{Id}. \quad (3.6)$$

By Theorem 3.1, $D_{(\Phi, \Psi)}^{(\nu)}$ is symmetric with respect to the weight $W^{(\nu)}$, and by Remark 3.2, $D_{(\Phi, \Psi)}^{(\nu)}$ commutes with J . By a straightforward calculation $D_{(\Phi, \Psi)}^{(\nu)}$ can be written as

$$D_{(\Phi, \Psi)}^{(\nu)} = \frac{d^2}{dx^2} \Phi^{(\nu)}(x)^* + \frac{d}{dx} \Psi^{(\nu)}(x)^*, \quad (3.7)$$

where $\Phi^{(\nu)}$ and $\Psi^{(\nu)}$ are explicit matrix-valued polynomials of degree 2 and 1;

$$\begin{aligned} \Phi^{(\nu)}(x) = & x^2 \sum_{i=0}^{2\ell} \frac{(\ell - i)^2 - (\ell + \nu)^2}{\ell^2} E_{i,i} + x \sum_{i=1}^{2\ell} \frac{(i - 1 - 2\ell)(2\ell - 2i + 1)}{2\ell^2} E_{i,i-1} + \\ & x \sum_{i=0}^{2\ell-1} \frac{(i + 1)(2\ell - 2i - 1)}{2\ell^2} E_{i,i+1} + \sum_{i=2}^{2\ell} \frac{(2\ell - i + 2)(2\ell - i + 1)}{4\ell^2} E_{i,i-2} + \\ & \sum_{i=0}^{2\ell} \frac{-i(2\ell - i + 1) - (2\ell - i)(i + 1) + 4(\ell + \nu)^2}{4\ell^2} E_{i,i} + \sum_{i=0}^{2\ell-2} \frac{(i + 2)(i + 1)}{4\ell^2} E_{i,i+2}, \end{aligned} \quad (3.8)$$

$$\begin{aligned} \Psi^{(\nu)}(x) = & -x \sum_{i=0}^{2\ell} \frac{(2\ell + 2\nu + 1)(\nu + i)(\nu + 2\ell - i)}{\ell^2} E_{i,i} - \\ & (\ell + \nu + \frac{1}{2}) \left(\sum_{i=1}^{2\ell} \frac{(i - 1 - 2\ell)(\nu + i - 1)}{\ell^2} E_{i,i-1} + \sum_{i=0}^{2\ell-1} \frac{(i + 1)(\nu + 2\ell - i - 1)}{\ell^2} E_{i,i+1} \right). \end{aligned} \quad (3.9)$$

Proposition 3.3. *Let $\nu > 0$. The weight matrix $W^{(\nu)}(x)$ and the matrix polynomials $\Phi^{(\nu)}(x)$ and $\Psi^{(\nu)}(x)$ satisfy $(W^{(\nu)}(x)\Phi^{(\nu)}(x))' = W^{(\nu)}(x)\Psi^{(\nu)}(x)$.*

Proof. Since $D_{(\Phi, \Psi)}^{(\nu)}$, as a combination of symmetric differential operators by Theorem 3.1, is symmetric, [6, Thm 3.1] applies, see also the proof of Theorem 3.1. Hence $(\Phi^{(\nu)}(x)^* W^{(\nu)}(x))'' - (\Psi^{(\nu)}(x)^* W^{(\nu)}(x))' = 0$ and the boundary condition

$$\lim_{x \rightarrow -1} (\Phi^{(\nu)}(x)^* W^{(\nu)}(x))' - \Psi^{(\nu)}(x)^* W^{(\nu)}(x) = 0$$

hold. Integrating the equation with respect to x gives $(\Phi^{(\nu)}(x)^* W^{(\nu)}(x))' = \Psi^{(\nu)}(x)^* W^{(\nu)}(x)$ using the boundary condition. Taking adjoints gives the result. \square

Theorem 3.4. *Let $\nu > 0$, $x \in (-1, 1)$, then $W^{(\nu+1)}(x) = c^{(\nu)} W^{(\nu)}(x) \Phi^{(\nu)}(x)$, where $c^{(\nu)} = \frac{(2\nu+1)(2\ell+\nu+1)\ell^2}{\nu(2\nu+2\ell+1)(2\ell+\nu)(\ell+\nu)}$.*

Proof. We prove the derived identity, where we use Proposition 3.3 in the left hand side,

$$c^{(\nu)} W^{(\nu)}(x) \Psi^{(\nu)}(x) = (W^{(\nu+1)}(x))'. \quad (3.10)$$

Arguing as in the beginning of the proof of Proposition 3.3 we also have the boundary condition $\lim_{x \rightarrow -1} \Phi^{(\nu)}(x)^* W^{(\nu)}(x) = 0$ from the symmetry of $D_{(\Phi, \Psi)}^{(\nu)}$, and this shows that it suffices to prove (3.10). The proof of (3.10) uses the expression in Definition 2.1 and properties of the Gegenbauer polynomials, and it is referred to Appendix B. \square

Now Corollary 3.5 is motivated by [4, Thm. 3.14]. Since we have to deal with the positive definite setting, we give the proof for completeness.

Corollary 3.5. *For $\nu > 0$, the sequence $(\frac{dP_n^{(\nu)}}{dx})_n$ is a sequence of matrix-valued orthogonal polynomials with respect to weight function $W^{(\nu+1)}$ on $[-1, 1]$. In particular, $\frac{dP_n^{(\nu)}}{dx}(x) = nP_{n-1}^{(\nu+1)}(x)$, $n \geq 1$.*

Proof. The leading coefficient $n\text{Id}$ of $\frac{dP_n}{dx}$ is non-singular, and for $k \in \mathbb{N}$ with $k < n - 1$ we get

$$\int_{-1}^1 \frac{dP_n^{(\nu)}}{dx}(x) W^{(\nu+1)}(x) x^k dx = -c^{(\nu)} \int_{-1}^1 P_n^{(\nu)}(x) \left((W^{(\nu)}(x) \Phi^{(\nu)}(x))' x^k + k W^{(\nu)}(x) \Phi^{(\nu)}(x) x^{k-1} \right) dx$$

using integration by parts and Theorem 3.4. Next we use Proposition 3.3 and considering the degrees we see that the integral is zero for all $k < n - 1$. \square

Denote the matrix-valued inner product for matrix-valued polynomials by (2.7), then the space of matrix-valued orthogonal polynomials with respect to (2.7) forms a pre-Hilbert C^* -module, see [16], with $M_{2\ell+1}(\mathbb{C})$ the corresponding (finite-dimensional) C^* -algebra. Note that we consider the pre-Hilbert C^* -module as a left module for $M_{2\ell+1}(\mathbb{C})$ and the inner product to be conjugate linear in the second variable, in contrast with [16]. Let $\mathcal{H}^{(\nu)}$ be the Hilbert C^* -module which is the completion [16, p. 4], then $\frac{d}{dx}: \mathcal{H}^{(\nu)} \rightarrow \mathcal{H}^{(\nu+1)}$ is an unbounded operator with dense domain and dense range, both being the matrix-valued polynomials in $\mathcal{H}^{(\nu)}$ and $\mathcal{H}^{(\nu+1)}$.

Lemma 3.6. *Let $\nu > 0$. Define the first order matrix-valued differential operator by*

$$(QT^{(\nu)})(x) = \frac{dQ}{dx}(x) (\Phi^{(\nu)}(x))^* + Q(x) (\Psi^{(\nu)}(x))^*,$$

then $\langle \frac{dP}{dx}, Q \rangle^{(\nu+1)} = -c^{(\nu)} \langle P, QT^{(\nu)} \rangle^{(\nu)}$ for matrix-valued polynomials P and Q .

Proof. Start with the left hand side, use integration by parts termwise noting that the boundary terms vanish using Definition 2.1 since $\nu + 1 - \frac{1}{2} > 0$, and next Theorem 3.4 and (3.10) to find

$$\langle \frac{dP}{dx}, Q \rangle^{(\nu+1)} = - \int_{-1}^1 P(x) \left((W^{(\nu)}(x) \Phi^{(\nu)}(x))' Q(x)^* + W^{(\nu)}(x) \Phi^{(\nu)}(x) \left(\frac{dQ}{dx}(x) \right)^* \right) dx.$$

Next use Proposition 3.3 to obtain the result. \square

In particular, Lemma 3.6 shows that, up to a constant, $T^{(\nu)}$ with domain the matrix-valued polynomials is a restriction of the adjoint of $\frac{d}{dx}: \mathcal{H}^{(\nu)} \rightarrow \mathcal{H}^{(\nu+1)}$, see [16, Ch. 9], so that $\frac{d}{dx}$ is a regular operator [16, p. 96].

Note that $PD_{(\Phi, \Psi)}^{(\nu)} = (\frac{dP}{dx}) T^{(\nu)}$, so that $c^{(\nu)} \langle PD_{(\Phi, \Psi)}^{(\nu)}, Q \rangle^{(\nu)} = -\langle \frac{dP}{dx}, \frac{dQ}{dx} \rangle^{(\nu+1)}$ which would give an alternative derivation of the symmetry of $D_{(\Phi, \Psi)}^{(\nu)}$ in case we would have a proof of Lemma 3.6 not using the symmetry of $D_{(\Phi, \Psi)}^{(\nu)}$, as it does now. We have opted to prove the

symmetry of $D^{(\nu)}$ and $E^{(\nu)}$ first, since the symmetry of $E^{(\nu)}$ cannot be obtained from the symmetry of $D_{(\Phi, \Psi)}^{(\nu)}$, cf. (3.7) and Remark 3.2.

We can now exploit Lemma 3.6 in order to obtain information on the monic matrix-valued Gegenbauer polynomials of (2.6).

Theorem 3.7. (i) *The squared norm $H_n^{(\nu)}$ in (2.6) is given by the diagonal matrix*

$$(H_n^{(\nu)})_{k,k} = \sqrt{\pi} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + 1)} \frac{\nu(2\ell + \nu + n)}{\nu + n} \frac{n! (\ell + \frac{1}{2} + \nu)_n (2\ell + \nu)_n (\ell + \nu)_n}{(2\ell + \nu + 1)_n (\nu + k)_n (2\ell + 2\nu + n)_n (2\ell + \nu - k)_n} \\ \times \frac{k! (2\ell - k)! (n + \nu + 1)_{2\ell}}{(2\ell)! (n + \nu + 1)_k (n + \nu + 1)_{2\ell - k}}$$

(ii) *The following Rodrigues formula holds:*

$$P_n^{(\nu)}(x) = G_n^{(\nu)} \frac{d^n}{dx^n} (W^{(\nu+n)}(x)) W^{(\nu)}(x)^{-1} \\ (G_n^{(\nu)})_{j,k} = \delta_{j,k} \frac{(-1)^n (\nu)_n (\ell + \nu + \frac{1}{2})_n (\ell + \nu)_n (2\ell + \nu)_n}{(\nu + \frac{1}{2})_n (\nu + k)_n (2\ell + \nu + 1)_n (2\ell + 2\nu + n)_n (2\ell + \nu - k)_n}.$$

Note that the Rodrigues formula has a compact nature and works for any size, and differs from the Rodrigues formula for the irreducible 2×2 -cases and $\nu = 1$ in [14, §8].

Proof. Since $D_{(\Phi, \Psi)}^{(\nu)}$, as the composition of $T^{(\nu)}$ after $\frac{d}{dx}$, has the monic matrix-valued Gegenbauer polynomials as eigenfunctions, and since $\frac{d}{dx}$ is a lowering operator by Corollary 3.5, we see that $(P_{n-1}^{(\nu+1)} T^{(\nu)})(x) = K_n^{(\nu)} P_n^{(\nu)}(x)$ where $K_n^{(\nu)}$ can be determined from the leading coefficients of $\Phi^{(\nu)}$ and $\Psi^{(\nu)}$. In particular, $T^{(\nu)}$ is a raising operator. Explicitly, $K_n^{(\nu)}$ is a diagonal invertible matrix;

$$(K_n^{(\nu)})_{k,k} = -\frac{(\nu + k)(2\ell + 2\nu + n)}{\ell^2} (2\ell + \nu - k)$$

and the eigenvalue matrix $\Lambda_n(D_{(\Phi, \Psi)}^{(\nu)}) = nK_n^{(\nu)}$. Now use Corollary 3.5 and Lemma 3.6 to see

$$n^2 \langle P_{n-1}^{(\nu+1)}, P_{n-1}^{(\nu+1)} \rangle^{(\nu+1)} = \langle \frac{dP_n^{(\nu)}}{dx}, \frac{dP_n^{(\nu)}}{dx} \rangle^{(\nu+1)} = -c^{(\nu)} \langle P_n^{(\nu)}, \frac{dP_n^{(\nu)}}{dx} T^{(\nu)} \rangle^{(\nu)} = \\ -c^{(\nu)} \langle P_n^{(\nu)}, P_n^{(\nu)} D_{(\Phi, \Psi)}^{(\nu)} \rangle^{(\nu)} = -c^{(\nu)} \langle P_n^{(\nu)}, P_n^{(\nu)} \rangle^{(\nu)} \Lambda_n(D_{(\Phi, \Psi)}^{(\nu)})^* = -\langle P_n^{(\nu)}, P_n^{(\nu)} \rangle^{(\nu)} n c^{(\nu)} (K_n^{(\nu)})^*.$$

Iterating, we can express the squared norm $\langle P_n^{(\nu)}, P_n^{(\nu)} \rangle^{(\nu)}$ in terms of $H_0^{(\nu+n)}$, see (2.8), and the product $\prod_{i=0}^{n-1} c^{(\nu+i)} K_{n-i}^{(\nu+i)}$. Note that the matrices commute, so we do not need to specify the order. This leads to (i).

For (ii), use Theorem 3.4 and (3.10) to see that $T^{(\nu)}$ can be written as $(QT^{(\nu)})(x) = (c^{(\nu)})^{-1} \frac{d}{dx} (Q(x) W^{(\nu+1)}(x)) W^{(\nu)}(x)^{-1}$, $-1 < x < 1$, which, by Lemma 3.6, preserves polynomials. Iterating shows

$$\left(\prod_{i=0}^{n-1} c^{(\nu+i)} \right) \left(((QT^{(\nu+n-1)}) \dots T^{(\nu+1)}) T^{(\nu)} \right)(x) = \frac{d^n}{dx^n} (Q(x) W^{(\nu+n)}(x)) W^{(\nu)}(x)^{-1}.$$

Now take $Q(x) = P_0^{(\nu+n)}(x) = 1$, so that the left hand side equals $\prod_{i=0}^{n-1} c^{(\nu+i)} K_{n-i}^{(\nu+i)} P_n^{(\nu)}(x)$. A calculation gives the diagonal matrix $G_n^{(\nu)}$. \square

Finally, note that the factorization of $D_{(\Phi, \Psi)}^{(\nu)}$ gives a Darboux transform. It is immediate from Corollary 3.5 and Proposition 3.3 that $\frac{d}{dx} \circ T^{(\nu)} : P \mapsto \frac{d(PT^{(\nu)})}{dx}$ is a matrix-valued differential operator symmetric with respect to the weight $W^{(\nu+1)}$ and which has the matrix-valued orthogonal polynomials $P_n^{(\nu+1)}$ as eigenfunctions. By a direct calculation we see that

$$\begin{aligned} \frac{d}{dx} \circ T^{(\nu)} &= \frac{d^2}{dx^2} \Phi^{(\nu)}(x)^* + \frac{d}{dx} \left(\frac{d}{dx} \Phi^{(\nu)}(x)^* + \Psi^{(\nu)}(x)^* \right) + \left(\frac{d}{dx} \Psi^{(\nu)}(x)^* \right) \\ &= (E^{(\nu+1)})^2 + (2\ell + 2)E^{(\nu+1)} + \left(\frac{\ell + \nu}{\ell} \right)^2 D^{(\nu+1)} + \left(\frac{\nu(\nu-1)(2\ell + \nu)(2\ell + \nu + 1)}{\ell^2} \right) \text{Id}, \end{aligned}$$

which should be compared to (3.6). In particular, $\frac{d}{dx} \circ T^{(\nu)}$ commutes with J .

4. EXPRESSION IN TERMS OF MATRIX-VALUED HYPERGEOMETRIC FUNCTIONS

In this section we link the matrix-valued Gegenbauer polynomials to Tirao's [21] matrix-valued hypergeometric series analogous to [14, §4]. In order to make this link we switch from the interval $[-1, 1]$ to $[0, 1]$ using $x = 1 - 2u$. Set

$$R_n^{(\nu)}(u) = (-1)^n 2^{-n} P_n^{(\nu)}(1 - 2u), \quad Z^{(\nu)}(u) = W_{\text{pol}}^{(\nu)}(1 - 2u) \quad (4.1)$$

hence the rescaled monic matrix-valued orthogonal polynomials $R_n^{(\nu)}$ satisfy

$$\int_0^1 R_n^{(\nu)}(u) Z^{(\nu)}(u) R_m^{(\nu)}(u)^* (u(1-u))^{\nu-1/2} du = \delta_{n,m} 2^{-2n-2\nu} H_n^{(\nu)}. \quad (4.2)$$

In this setting Theorem 3.1 gives the following corollary.

Corollary 4.1. *Let $\tilde{D}^{(\nu)}$ and $\tilde{E}^{(\nu)}$ be the matrix-valued differential operators*

$$\tilde{D}^{(\nu)} = u(1-u) \frac{d^2}{du^2} + \frac{d}{du} (\tilde{C}^{(\nu)} - u\tilde{U}^{(\nu)}) - \tilde{V}^{(\nu)}, \quad \tilde{E}^{(\nu)} = \frac{d}{du} (u\tilde{B}_1^{(\nu)} + \tilde{B}_0^{(\nu)}) + \tilde{A}_0^{(\nu)},$$

where $\tilde{C}^{(\nu)}$, $\tilde{U}^{(\nu)}$, $\tilde{V}^{(\nu)}$, $\tilde{B}_0^{(\nu)}$, $\tilde{B}_1^{(\nu)}$ and $\tilde{A}_0^{(\nu)}$ are given by

$$\begin{aligned} \tilde{C}^{(\nu)} &= - \sum_{i=0}^{2\ell} \frac{(2\ell-i)}{2} E_{i,i+1} + \sum_{i=0}^{2\ell} \frac{(2\ell+2\nu+1)}{2} E_{i,i} - \sum_{i=0}^{2\ell} \frac{i}{2} E_{i,i-1}, \quad \tilde{U}^{(\nu)} = (2\ell+2\nu+1)I, \\ \tilde{V}^{(\nu)} &= - \sum_{i=0}^{2\ell} i(2\ell-i) E_{i,i} + (\nu-1)(2\ell+\nu+1)I \\ \tilde{B}_0^{(\nu)} &= - \sum_{i=0}^{2\ell} \frac{(2\ell-i)}{4\ell} E_{i,i+1} + \sum_{i=0}^{2\ell} \frac{(\ell-i)}{2\ell} E_{i,i} + \sum_{i=0}^{2\ell} \frac{i}{4\ell} E_{i,i-1}, \quad \tilde{B}_1^{(\nu)} = - \sum_{i=0}^{2\ell} \frac{(\ell-i)}{\ell} E_{i,i}, \\ \tilde{A}_0^{(\nu)} &= \sum_{i=0}^{2\ell} \left(\frac{(\ell+1)(i-2\ell)}{\ell} - (\nu-1) \frac{(\ell-i)}{\ell} \right) E_{i,i}. \end{aligned}$$

Then $\tilde{D}^{(\nu)}$ and $\tilde{E}^{(\nu)}$ are symmetric with respect to the weight $(u(1-u))^{\nu-1/2}Z^{(\nu)}(u)$, and $\tilde{D}^{(\nu)}$ and $\tilde{E}^{(\nu)}$ commute. Moreover, for every integer $n \geq 0$, $R_n^{(\nu)}\tilde{D}^{(\nu)} = \Lambda_n(\tilde{D}^{(\nu)})R_n^{(\nu)}$, and $R_n^{(\nu)}\tilde{E}^{(\nu)} = \Lambda_n(\tilde{E}^{(\nu)})R_n^{(\nu)}$ with

$$\begin{aligned}\Lambda_n(\tilde{D}^{(\nu)}) &= ((\nu-1)(2\ell+\nu+1) - n(2\ell+2\nu+n))\text{Id} - \sum_{i=0}^{2\ell} i(2\ell-i)E_{i,i}, \\ \Lambda_n(\tilde{E}^{(\nu)}) &= -(n+\nu+2\ell+1)\text{Id} - (n+\nu+\ell) \sum_{i=0}^{2\ell} \frac{i}{\ell} E_{i,i}.\end{aligned}$$

As in [15, §4] we describe the rows of $R_n^{(\nu)}$ in terms of matrix-valued hypergeometric functions [21]. We first need to study the $\mathbb{C}^{2\ell+1}$ -valued polynomial solutions of $P\tilde{D}^{(\nu)} = \lambda P$, $\lambda \in \mathbb{C}$. In order to avoid technical problems we consider

$$D_\alpha^{(\nu)} = \tilde{D}^{(\nu)} + \alpha\tilde{E}^{(\nu)} = u(1-u)\frac{d^2}{du^2} + \frac{d}{du}(C_\alpha^{(\nu)} - uU_\alpha^{(\nu)}) - V_\alpha^{(\nu)}, \quad \alpha \in \mathbb{R}, \quad (4.3)$$

where $C_\alpha^{(\nu)} = \tilde{C}^{(\nu)} + \alpha\tilde{B}_0^{(\nu)}$, $U_\alpha^{(\nu)} = \tilde{U}^{(\nu)} - \alpha\tilde{B}_1^{(\nu)}$ and $V_\alpha^{(\nu)} = \tilde{V}^{(\nu)} - \alpha\tilde{A}_0^{(\nu)}$. It follows from Corollary 4.1 that

$$R_n^{(\nu)}D_\alpha^{(\nu)} = \Lambda_n(D_\alpha^{(\nu)})R_n^{(\nu)}, \quad \Lambda_n(D_\alpha^{(\nu)}) = -n^2 - n(\tilde{U}_\alpha^{(\nu)} - 1) - \tilde{V}_\alpha^{(\nu)}, \quad \text{for all } n \in \mathbb{N}_0. \quad (4.4)$$

We denote by $\lambda_n^\alpha(j)$ the j -th diagonal entry of $\Lambda_n(D_\alpha^{(\nu)})$, i.e.

$$\begin{aligned}\lambda_n^\alpha(j) &= -n^2 - n \frac{(2\ell(\ell+\nu) + \alpha(\ell-j) - \ell)}{\ell} \\ &\quad - \frac{(2\ell-j)(\alpha(\ell+1) - \ell j)}{\ell} + (\nu-1) \frac{\ell(2\ell+\nu+1) - \alpha(\ell-j)}{\ell}.\end{aligned}$$

It follows from (4.4) that the i -th row of $R_n^{(\nu)}$ is a solution to

$$u(1-u)F''(u) + F'(u)(C_\alpha^{(\nu)} - uU_\alpha^{(\nu)}) - F(u)(V_\alpha^{(\nu)} + \lambda) = 0, \quad \lambda = (\Lambda_n(D_\alpha^{(\nu)}))_{i,i}, \quad (4.5)$$

an instance of the matrix-valued hypergeometric equation [21]. In order to be able to apply Tirao's [21] approach, and to have the rows of $R_n^{(\nu)}$ defined by this solution, we need Lemma 4.2.

Lemma 4.2. (i) The eigenvalues of $C_\alpha^{(\nu)}$ are $(2j+2\nu+1)/2$, $j \in \{0, \dots, 2\ell\}$. In particular, $\sigma(C_\alpha^{(\nu)}) \cap \{-\mathbb{N}\} = \emptyset$.

(ii) Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $(j, n) = (i, m) \in \{0, 1, \dots, 2\ell\} \times \mathbb{N}$ if and only if $\lambda_n^\alpha(j) = \lambda_m^\alpha(i)$.

Proof. Part (i) is proved as [15, Lem. 4.3], using the same Krawtchouk polynomials, and there is only a shift in the eigenvalues. Part (ii) is proved analogously to [15, Lem. 4.4]. \square

Since the eigenvalues of $C_\alpha^{(\nu)}$ are not in $-\mathbb{N}$, for any $F_0 \in \mathbb{C}^{2\ell+1}$, a (row-)vector-valued solution to (4.5) is given by

$$F(u) = \left({}_2H_1 \left(\begin{matrix} (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda \\ (C_\alpha^{(\nu)})^t \end{matrix}; u \right) F_0 \right)^t \quad (4.6)$$

where the matrix-valued hypergeometric function ${}_2H_1$, defined as the power series

$${}_2H_1 \left(\begin{matrix} U, V \\ C \end{matrix} ; z \right) = \sum_{i=0}^{\infty} \frac{z^i}{i!} [C, U, V]_i, \quad (4.7)$$

$$[C, U, V]_0 = \text{Id}, \quad [C, U, V]_{i+1} = (C + i)^{-1} (i^2 + i(U - 1) + V) [C, U, V]_i,$$

converges for $|z| < 1$ in $M_{2\ell+1}(\mathbb{C})$.

So (4.6) is valid and this gives a series representation for the rows of the monic polynomial $R_n^{(\nu)}$. Since each row is polynomial, the series has to terminate and there exists $n \in \mathbb{N}$ so that $[(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_{n+1}$ is singular and $0 \neq P_0 \in \text{Ker} \left([(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_{n+1} \right)$.

Suppose that n is the least integer for which $[(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_{n+1}$ is singular, i.e. $[(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_i$ is regular for all $i \leq n$. Since

$$\begin{aligned} & [(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_{n+1} \\ &= ((C_\alpha^{(\nu)})^t + n)^{-1} (n^2 + n((U_\alpha^{(\nu)})^t - 1) + (V_\alpha^{(\nu)})^t + \lambda) [(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_n \end{aligned} \quad (4.8)$$

and since the matrix $(C_\alpha^{(\nu)} + n)$ is invertible by Lemma 4.2(i), $[(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_{n+1}$ is a singular matrix if and only if the diagonal matrix

$$\begin{aligned} M_n^\alpha(\lambda) &= (n^2 + n((U_\alpha^{(\nu)})^t - 1) + V_\alpha^t + \lambda) \\ &= (n^2 + n(U_\alpha^{(\nu)} - 1) + V_\alpha^{(\nu)} + \lambda) = \lambda - \Lambda_n(D_\alpha^{(\nu)}) \end{aligned} \quad (4.9)$$

is singular. Note that the diagonal entries of $M_n^\alpha(\lambda)$ are of the form $\lambda - \lambda_n^\alpha(j)$, so that $M_n^\alpha(\lambda)$ is singular if and only if $\lambda = \lambda_n^\alpha(j)$ for some $j \in \{0, 1, \dots, 2\ell\}$. Because of Lemma 4.2(ii) the value of λ and corresponding n (and j) is uniquely determined for α irrational.

Assume α irrational, so that $M_n^\alpha(\lambda_n^\alpha(i))$ is singular if and only if $n = m$. So in the series (4.6) the matrix $[(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_{n+1}$ is singular and $[(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_i$ is non-singular for $0 \leq i \leq n$. Furthermore, the kernel of $[(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_{n+1}$ is one-dimensional if and only if $\lambda = \lambda_n^\alpha(i)$, $i \in \{0, 1, \dots, 2\ell\}$. In case $\lambda = \lambda_n^\alpha(i)$ we see that (4.6) is a (row-)vector-valued polynomial for

$$P_0 = [(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda_n^\alpha(i)]_n^{-1} e_i$$

determined uniquely up to a scalar, where e_i , $i \in \{0, 1, \dots, 2\ell\}$, is a standard basis vector in $\mathbb{C}^{2\ell+1}$.

This leads to the main result of this section, expressing the monic polynomials $R_n^{(\nu)}$ as a matrix-valued hypergeometric function.

Theorem 4.3. *The monic matrix-valued orthogonal polynomials are given by*

$$(R_n^{(\nu)}(u))_{i,j} = \left({}_2H_1 \left(\begin{matrix} (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda_n^\alpha(i) \\ (C_\alpha^{(\nu)})^t \end{matrix} ; u \right) n! [(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda_n^\alpha(i)]_n^{-1} e_i \right)_j^t$$

for all $\alpha \in \mathbb{R}$.

In particular, the right hand side is independent of α .

Remark 4.4. Let $F(u) = {}_2H_1\left(\begin{smallmatrix} U, V \\ C \end{smallmatrix}; u\right) F(0)$, then $F'(u) = {}_2H_1\left(\begin{smallmatrix} U+2, V+U \\ C+1 \end{smallmatrix}; u\right) F'(0)$, since $[C, U, V]_{n+1} = [C+1, U+2, V+U]_n [C, U, V]_1$ and $[C, U, V]_1 = C^{-1}V$, and $F'(0) = C^{-1}VF(0)$. So we can use this in Theorem 4.3 to calculate $\left(\frac{dR_n^{(\nu)}}{du}(u)\right)_{ij}$ in terms of matrix-valued hypergeometric functions. On the other hand, using Corollary 3.5 and (4.1) we see that $\left(\frac{dR_n^{(\nu)}}{du}(u)\right)_{i,j} = n(R_{n-1}^{(\nu+1)}(u))_{i,j}$ can be expressed using Theorem 4.3 again. It can be checked directly, by using the explicit expressions in (4.3), that the matrices occurring as parameters in the two expressions as ${}_2H_1$ -series coincide.

5. THREE-TERM RECURRENCE RELATION

Since the monic matrix-valued polynomials $R_n^{(\nu)}$ are orthogonal, there is a three-term recurrence relation of the form

$$uR_n^{(\nu)}(u) = R_{n+1}^{(\nu)}(u) + X_n^{(\nu)}R_n^{(\nu)}(u) + Y_n^{(\nu)}R_{n-1}^{(\nu)}(u), \quad n \geq 0,$$

where $R_{-1}^{(\nu)} = 0$ and $X_n^{(\nu)}, Y_n^{(\nu)} \in M_{2\ell+1}(\mathbb{C})$ are matrices depending on n and not on u . In order to obtain the coefficients $X_n^{(\nu)}, Y_n^{(\nu)}$ explicitly, we exploit the explicit expression of $R_n^{(\nu)}$ in terms of Tirao's matrix-valued hypergeometric function of Theorem 4.3. For this we need the following lemma, see [15, Lem. 5.1].

Lemma 5.1. *Let $\{R_n^{(\nu)}\}_{n \geq 0}$ be a sequence of monic orthogonal polynomials and write $R_n^{(\nu)}(u) = \sum_{k=0}^n R_{n,k}^{(\nu)} u^k$, $R_k^{(\nu)} \in M_{2\ell+1}(\mathbb{C})$, and $R_{n,n}^{(\nu)} = \text{Id}$. Then the coefficients $X_n^{(\nu)}, Y_n^{(\nu)}$ of the three-term recurrence relation are given by*

$$X_n^{(\nu)} = R_{n,n-1}^{(\nu)} - R_{n+1,n}^{(\nu)}, \quad Y_n^{(\nu)} = R_{n,n-2}^{(\nu)} - R_{n+1,n-1}^{(\nu)} - X_n^{(\nu)}R_{n,n-1}^{(\nu)}.$$

In [15] we needed $R_{n,n-1}^{(\nu)}$ and $R_{n,n-2}^{(\nu)}$, but since we already have established the norm matrix $H_n^{(\nu)}$ in Theorem 3.7, see (4.1), thanks to the raising and lowering operators, we can use $Y_n^{(\nu)} = \frac{1}{4}H_n^{(\nu)}(H_{n-1}^{(\nu)})^{-1}$ to calculate $Y_n^{(\nu)}$ as a diagonal matrix, see Theorem 5.3 for the explicit expression. So we need only to calculate X_n , for which we require the coefficients $R_{n,n-1}^{(\nu)}$. This can be obtained directly from the explicit expression in Theorem 4.3.

Lemma 5.2. *Let $R_n^{(\nu)}$ be matrix-valued orthogonal polynomial as in Theorem 4.3, then*

$$R_{n,n-1}^{(\nu)} = \sum_{j=0}^{2\ell} \frac{jn}{4(j+n+\nu-1)} E_{j,j-1} - \sum_{j=0}^{2\ell} \frac{n}{2} E_{j,j} + \sum_{j=0}^{2\ell} \frac{n(2\ell-j)}{4(2\ell+n+\nu-j-1)} E_{j,j+1}$$

Proof. We compute $R_{n,n-1}^{(\nu)}$ by considering the coefficient of u^{n-1} in Theorem 4.3. Using the recursive definition of $[C_\alpha^{(\nu)}, U_\alpha^{(\nu)}, V_\alpha^{(\nu)} + \lambda_n^\alpha(i)]_n$ we obtain

$$(R_{n,n-1}^{(\nu)})_{i,j} = (e_i^t(C_\alpha^{(\nu)} + n - 1)M_{n-1}^\alpha(\lambda_n^\alpha(i))^{-1})_j.$$

Observe that the matrix $M_{n-1}^\alpha(\lambda_n^\alpha(i))$ is invertible for irrational α . Now the lemma follows by a straightforward computation. \square

Theorem 5.3. *For any $\ell \in \frac{1}{2}\mathbb{N}$ the monic orthogonal polynomials $R_n^{(\nu)}$ satisfy the following three term recurrence relation*

$$uR_n^{(\nu)}(u) = R_{n+1}^{(\nu)}(u) + X_n^{(\nu)}R_n^{(\nu)}(u) + Y_n^{(\nu)}R_{n-1}^{(\nu)}(u), \quad (5.1)$$

where the matrices $X_n^{(\nu)}, Y_n^{(\nu)}$ are given by

$$X_n^{(\nu)} = \sum_{j=0}^{2\ell} \left[\frac{-j(j+\nu-1)}{4(j+n+\nu-1)(j+n+\nu)} E_{j,j-1} + \frac{E_{j,j}}{2} - \frac{(2\ell-j)(2\ell-j+\nu-1)}{4(2\ell-j+n+\nu-1)(2\ell+n-j+\nu)} E_{j,j+1} \right],$$

$$Y_n^{(\nu)} = \sum_{j=0}^{2\ell} \frac{n(n+\nu-1)(2\ell+n+\nu)(2\ell+n+2\nu-1)}{16(2\ell+n+\nu-j-1)(2\ell+n+\nu-j)(j+n+\nu-1)(j+n+\nu)} E_{j,j}.$$

Proof. The statement for $X_n^{(\nu)}$ is a direct consequence of Lemmas 5.2 and 5.1. We use $Y_n^{(\nu)} = \frac{1}{4}H_n^{(\nu)}(H_{n-1}^{(\nu)})^{-1}$ for the other expression. \square

The monic matrix-valued Gegenbauer polynomials $P_n^{(\nu)}$ defined in (2.6) satisfy

$$xP_n^{(\nu)}(x) = P_{n+1}^{(\nu)}(x) + (1 - 2X_n^{(\nu)})P_n^{(\nu)}(x) + 4Y_n^{(\nu)}P_{n-1}^{(\nu)}(x) \quad (5.2)$$

with the notation as in Theorem 5.3. Note that in the limit $n \rightarrow \infty$ the coefficients become constant; $\lim_{n \rightarrow \infty} (1 - 2X_n^{(\nu)}) = 0$, $\lim_{n \rightarrow \infty} 4Y_n^{(\nu)} = \frac{1}{4}$.

Taking derivatives and using Corollary 3.5 gives the identity

$$P_n^{(\nu)}(x) + nxP_{n-1}^{(\nu+1)}(x) = (n+1)P_n^{(\nu+1)}(x) + n(1 - 2X_n^{(\nu)})P_{n-1}^{(\nu+1)}(x) + 4(n-1)Y_n^{(\nu)}P_{n-2}^{(\nu+1)}(x) \quad (5.3)$$

for $n \geq 1$ and with the convention $P_{-1}^{(\nu+1)}(x) = 0$, see also [4, Thm. 3.14].

Combining with the three-term recurrence for $xP_{n-1}^{(\nu+1)}(x)$ gives

$$P_n^{(\nu)}(x) = P_n^{(\nu+1)}(x) + 2(X_{n-1}^{(\nu+1)} - X_n^{(\nu)})P_{n-1}^{(\nu+1)}(x) + 4((n-1)Y_n^{(\nu)} - nY_{n-1}^{(\nu+1)})P_{n-2}^{(\nu+1)}(x). \quad (5.4)$$

6. THE MATRIX-VALUED ORTHOGONAL POLYNOMIALS RELATED TO GEGENBAUER AND RACA POLYNOMIALS

The matrix entries of $P_n^{(\nu)}$ can be expressed in terms of Gegenbauer and Racah polynomials, and we give the precise connection in Corollary 6.3. This extends the case $\nu = 1$ of [15, §6], and incorporates the recent results of Cagliero and Koornwinder [3].

We switch to $D_0^{(\nu)} = \tilde{D}^{(\nu)} - 2\ell\tilde{E}^{(\nu)}$, since this will lead to a diagonalisable operator after conjugation;

$$\begin{aligned} D_0^{(\nu)} &= u(1-u)\frac{d^2}{du^2} + \left(\frac{d}{du}\right)(K_0^1 - uK_1^1) + K_0, \\ K_1^1 &= \sum_{i=0}^{2\ell} (2i + 2\nu + 1) E_{i,i}, \quad K_0^1 = -\sum_{i=0}^{2\ell} i E_{i,i-1} + \sum_{i=0}^{2\ell} \frac{(2i + 2\nu + 1)}{2} E_{i,i}, \\ K_0 &= \sum_{i=0}^{2\ell} ((2\ell - i)(2\ell + i + 2) - (\nu - 1)(\nu + 2i + 1)) E_{i,i}. \end{aligned} \quad (6.1)$$

Note that $D_0^{(\nu)}$ is a matrix-valued hypergeometric differential operator in the sense of Tirao [21]. We conjugate this differential operator by $M^{(\nu)}(u) = L^{(\nu)}(1-2u)$, where $L^{(\nu)}$ is the L-part of the LDU-decomposition of the weight of Theorem 2.3. Going through to the calculations as in [15, §6], we obtain Proposition 6.1. As in [15, §6] this is the only linear combination of $\tilde{D}^{(\nu)}$ and $\tilde{E}^{(\nu)}$ leading to a diagonalisable differential operator after conjugation with $M^{(\nu)}$.

Proposition 6.1. *The differential operator $\mathcal{D}^{(\nu)} = (M^{(\nu)})^{-1}D_0^{(\nu)}M^{(\nu)}$ is the diagonal differential operator*

$$\mathcal{D}^{(\nu)} = u(1-u)\frac{d^2}{du^2} + \left(\frac{d}{du}\right)T_1(u) + T_0,$$

where $T_1(u) = \frac{1}{2}T_1^1 - uT_1^1$,

$$T_1^1 = \sum_{k=0}^{2\ell} (2k + 2\nu + 1)E_{k,k}, \quad T_0 = \sum_{k=0}^{2\ell} (2\ell - k - \nu + 1)(2\ell + k + \nu + 1)E_{k,k}.$$

Moreover, $\mathcal{R}_n^{(\nu)}(u) = R_n^{(\nu)}(u)M^{(\nu)}(u)$ satisfies

$$\mathcal{R}_n^{(\nu)} \mathcal{D}^{(\nu)} = \Lambda_n(\mathcal{D}^{(\nu)}) \mathcal{R}_n^{(\nu)}, \quad \Lambda_n(\mathcal{D}^{(\nu)}) = \Lambda_n(\tilde{D}^{(\nu)}) - 2\ell \Lambda_n(\tilde{E}^{(\nu)}).$$

The proof of Proposition 6.1 is completely analogous to the proof of [15, Prop. 6.1].

Since $R_n^{(\nu)}$ and $M^{(\nu)}$ are matrix-valued polynomials, Proposition 6.1 implies that $(\mathcal{R}_n^{(\nu)}(u))_{k,j}$ is a polynomial solution to the hypergeometric differential operator

$$u(1-u)f''(u) + ((j + \nu + \frac{1}{2}) - u(2j + 2\nu + 1))f'(u) - (j - k - n)(j + n + k + 2\nu)f(u) = 0.$$

Since the polynomial solutions are unique up to a constant we find

$$(\mathcal{R}_n^{(\nu)}(u))_{k,j} = c_{k,j}^{(\nu)}(n) {}_2F_1\left(\begin{matrix} j - k - n, n + k + j + 2\nu \\ j + \frac{1}{2} + \nu \end{matrix}; u\right).$$

Note that the ${}_2F_1$ -series gives a Gegenbauer polynomial $C_{n+k-j}^{(\nu+j+1)}(1-2u)$ scaled to $[0, 1]$.

The matrix-valued differential operators $\tilde{D}^{(\nu)}$ and $\tilde{E}^{(\nu)}$ commute by Corollary 4.1. Hence the matrix-valued differential operator $\mathcal{E}^{(\nu)} = (M^{(\nu)})^{-1} \tilde{E}^{(\nu)} M^{(\nu)}$ satisfies

$$\mathcal{E}^{(\nu)} \mathcal{R}_n^{(\nu)} = \Lambda_n(\mathcal{E}^{(\nu)}) \mathcal{R}_n^{(\nu)}, \quad \Lambda_n(\mathcal{E}^{(\nu)}) = \Lambda_n(\tilde{E}^{(\nu)}), \quad \mathcal{E}^{(\nu)} \mathcal{D}^{(\nu)} = \mathcal{D}^{(\nu)} \mathcal{E}^{(\nu)} \quad (6.2)$$

An explicit calculation gives

$$\begin{aligned} \mathcal{E}^{(\nu)} &= \frac{d}{du} S_1(u) + S_0(u), \\ S_1(u) &= u(1-u) \sum_{i=0}^{2\ell} \frac{i(i+2\nu-2)(2\nu+i+2\ell-1)}{\ell(2\nu+2i-1)(2\nu+2i-3)} E_{i,i-1} + \sum_{i=0}^{2\ell} \frac{(2\ell-i)}{4\ell} E_{i,i+1}, \\ S_0(u) &= (1-2u) \sum_{i=0}^{2\ell} \frac{i(2\nu+i-2)(2\nu+i+2\ell-1)}{2\ell(2\nu+2i-3)} E_{i,i-1} \\ &\quad + \sum_{i=0}^{2\ell} \frac{i(2\nu+i-1) - 4\ell(\ell+1) - 2\ell(\nu-1)}{2\ell} E_{i,i}. \end{aligned} \quad (6.3)$$

By Tirao [21] analytic (row-)vector valued eigenfunctions of $\mathcal{D}^{(\nu)}$ for an eigenvalue λ are completely determined by the initial (row-) vector value at 0, i.e. by an element of $\mathbb{C}^{2\ell+1}$. Since $\mathcal{E}^{(\nu)}$ commutes with $\mathcal{D}^{(\nu)}$, $\mathcal{E}^{(\nu)}$ preserves the eigenspaces, and hence acts on the space $\mathbb{C}^{2\ell+1}$ of initial values. This gives rise to a linear operator $N(\lambda)$ on the space of initial values determining the eigenfunctions of $\mathcal{D}^{(\nu)}$ for eigenvalue λ , see [15, §6] for the approach for the special case $\nu = 1$.

Write an eigenfunction F of $\mathcal{D}^{(\nu)}$ for eigenvalue λ as, see [21] and cf. §4,

$$F_j(u) = \left({}_2H_1 \left(\begin{matrix} T_1^1, \lambda - T_0 \\ \frac{1}{2}T_1^1 \end{matrix}; u \right) F(0)^t \right)_j,$$

so that $\frac{dF_j}{du}(0) = F(0)(\lambda - T_0)(\frac{1}{2}T_1^1)^{-1}$ by construction of the ${}_2H_1$ -series, see [21]. Now (6.3) shows that

$$N(\lambda) = (\lambda - T_0)(\frac{1}{2}T_1^1)^{-1} S_1(0) + S_0(0)$$

acting from the right on row-vectors from $\mathbb{C}^{2\ell+1}$.

The k -th row $((\mathcal{R}_n^{(\nu)})_{k,j})_{j=0}^{2\ell}$ is an eigenfunction of $\mathcal{D}^{(\nu)}$ for the eigenvalue $\lambda_n(k) = \Lambda_n(\mathcal{D}^{(\nu)})_{k,k}$ by Proposition 6.1. On the other hand, the k -th row of $\mathcal{R}_n^{(\nu)}$ is an eigenfunction of $\mathcal{E}^{(\nu)}$ for the eigenvalue $\mu_n(k) = \Lambda_n(\mathcal{E}^{(\nu)})_{k,k}$. Since $((\mathcal{R}_n^{(\nu)}(0))_{k,j})_{j=0}^{2\ell} = (c_{k,j}^{(\nu)}(n))_{j=0}^{2\ell}$ the row-vector $c_k^{(\nu)} = (c_{k,j}^{(\nu)}(n))_{j=0}^{2\ell}$ satisfies $c_k^{(\nu)} N(\lambda_n(k)) = \mu_n(k) c_k^{(\nu)}$. Explicitly,

$$\begin{aligned} & - \frac{(i+k+n+2\nu-1)(i-k-n-1)(2\ell-i+1)}{(2i+2\nu-1)} c_{k,i-1}^{(\nu)}(n) \\ & + (i(i+2\nu-1) - 4\ell(\ell+1) - 2\ell(\nu-1)) c_{k,i}^{(\nu)}(n) + \frac{(i+1)(i+2\nu-1)(2\ell+i+2\nu)}{(2i+2\nu-1)} c_{k,i+1}^{(\nu)}(n) \\ & = (-2n(\ell-k) + (2\ell+2)(k-2\ell) - 2(\nu-1)(\ell-k)) c_{k,i}^{(\nu)}(n), \quad (6.4) \end{aligned}$$

This is a finite three-term recurrence relation, which can be solved explicitly in terms of Racah polynomials, see [22], [13],

$$\begin{aligned} c_{k,i}^{(\nu)} &= c_{k,0}^{(\nu)} (-1)^i \frac{(-2\ell)_i (-n-k)_i}{i! (2\nu+2\ell)_i} R_k(\lambda(i); -2\ell-1, -n-k-\nu, \nu-1, \nu-1) \\ &= c_{k,0}^{(\nu)} (-1)^i \frac{(-2\ell)_i (-n-k)_i}{i! (2\nu+2\ell)_i} {}_4F_3 \left(\begin{matrix} -i, i+2\nu-1, -k, -n-\nu-2\ell \\ \nu, -n-k, -2\ell \end{matrix}; 1 \right) \end{aligned} \quad (6.5)$$

which corresponds nicely with [15] for $\nu = 1$.

Switching to the variable x , we find

$$(\mathcal{P}_n^{(\nu)})_{k,j} = (P_n^{(\nu)}(x)L^{(\nu)}(x))_{k,j} = (-2)^n c_{k,j}^{(\nu)}(n) \frac{(n+k-j)!}{(2j+2\nu)_{n+k-j}} C_{n+k-j}^{(\nu+j)}(x),$$

Then the orthogonality relations (2.6) and the orthogonality relations for the (scalar-valued) Gegenbauer polynomials, see e.g. (A.1), imply

$$\delta_{n,m}(H_n)_{k,i} = \delta_{n+k,m+i} (-2)^{n+m} \sum_{j=0}^{2\ell \wedge (n+k)} c_{k,j}^{(\nu)}(n) \overline{c_{i,j}^{(\nu)}(m)} t_j^{(\nu)} \frac{(n+k-j)!}{(2\nu+2j)_{n+k-j}} \frac{\sqrt{\pi}\Gamma(\nu+j+1/2)}{(n+k+\nu)\Gamma(\nu+j)}.$$

It follows from (6.5)

$$\begin{aligned} \delta_{n,m}(H_n)_{k,k} &= (-2)^{-n-m} \frac{(n+k+\nu)\Gamma(\nu)\Gamma(n+k+2\nu)}{(2\ell+\nu)(n+k)!\Gamma(\nu+1/2)\sqrt{\pi}\Gamma(2\nu)} \\ &= |c_{k,0}^{(\nu)}(n)|^2 \sum_{j=0}^{2\ell \wedge (n+k)} \frac{(-2\ell)_j (-n-k)_j (2\nu-1)_j (\nu+1/2)_j}{j! (2\ell+2\nu)_j (n+k+2\nu)_j (\nu-1/2)_j} \\ &\quad \times R_k(\lambda(j); -2\ell-1, -k-n-\nu, \nu-1, \nu-1) R_{k+n-m}(\lambda(j); -2\ell-1, -k-n-\nu, \nu-1, \nu-1). \end{aligned} \quad (6.6)$$

which corresponds to the orthogonality relations for the corresponding Racah polynomials, see [22], [13]. From this we find that the sum in (6.6) equals

$$\delta_{n,m} M \frac{(-2\ell-n-\nu)_k (-2\ell-n-k-2\nu+1)_k (-2\ell-\nu+1)_k (-n-k-\nu+1)_k k!}{(-2\ell-n-k-\nu+1)_{2k} (-2\ell)_k (-n-k)_k (\nu)_k},$$

where

$$M = \frac{(n+k+\nu)_{2\ell} (2\nu)_{2\ell}}{(n+k+2\nu)_{2\ell} (\nu)_{2\ell}} = \frac{(2\ell+\nu)_{n+k} (2\nu)_{n+k}}{(2\ell+2\nu)_{n+k} (\nu)_{n+k}}.$$

Hence

$$|c_{k,0}^{(\nu)}(n)|^2 = \frac{4^{-2n} (\nu)_n^2 (2\ell+2\nu)_n^2}{(k+\nu)_n^2 (2\ell+\nu-k)_n^2}.$$

If we take the $(k, 0)$ -entry of the three term recurrence relation for $\mathcal{R}_n^{(\nu)}(u)$ we obtain a polynomial identity in u . If we take the leading coefficient it gives

$$c_{k,0}^{(\nu)}(n+1) = -c_{k,0}^{(\nu)}(n) \frac{(n+k+2\nu)}{4(n+k+\nu)} + c_{k+1,0}^{(\nu)}(n) \frac{(2\ell-k)(2\ell-k+\nu-1)(n+k+\nu)}{4(2\ell-k+n+\nu-1)(2\ell+n-k+\nu)}.$$

If we plug in $c_{k,0}^{(\nu)}(n) = \text{sgn}(c_{k,0}^{(\nu)}(n)) |c_{k,0}^{(\nu)}(n)|$ in the equation above, we obtain

$$\begin{aligned} & (\nu + n)(2\ell + 2\nu + n) \text{sgn}(c_{k,0}^{(\nu)}(n+1)) \\ &= -(n + k + 2\nu)(2\ell + n + \nu - k) \text{sgn}(c_{k,0}^{(\nu)}(n)) + (k + \nu)(2\ell - k) \text{sgn}(c_{k+1,0}^{(\nu)}(n)). \end{aligned}$$

It then follows that $\text{sgn}(c_{k,0}^{(\nu)}(n)) = \text{sgn}(c_{k+1,0}^{(\nu)}(n)) = -\text{sgn}(c_{k,0}^{(\nu)}(n+1))$. This implies that $\text{sgn}(c_{k,0}^{(\nu)}(n)) = (-1)^n$.

Theorem 6.2. *The polynomials $\mathcal{R}_n^{(\nu)}$ are given by*

$$\begin{aligned} (\mathcal{R}_n^{(\nu)}(u))_{k,j} &= c_{k,0}^{(\nu)}(n) (-1)^j \frac{(-2\ell)_j (-n-k)_j}{j! (2\nu + 2\ell)_j} \\ &\times {}_4F_3 \left(\begin{matrix} -j, j+2\nu-1, -k, -n-\nu-2\ell \\ \nu, -n-k, -2\ell \end{matrix} ; 1 \right) {}_2F_1 \left(\begin{matrix} j-k-n, n+k+j+2\nu \\ j+\frac{1}{2}+\nu \end{matrix} ; u \right). \end{aligned}$$

where the constant $c_{k,0}^{(\nu)}(n)$ is given by

$$c_{k,0}^{(\nu)}(n) = \frac{(-1)^n 4^{-n} (\nu)_n (2\ell + 2\nu)_n}{(k + \nu)_n (2\ell + \nu - k)_n}.$$

Note that Theorem 6.2 extends [15, Thm. 6.2]. Switching back to $P_n^{(\nu)}$ and using the inverse of $L^{(\nu)}(x)$ in (2.4) due Cagliero and Koornwinder [3], we get the following corollary.

Corollary 6.3. *Using the notation of Theorem 6.2 the monic matrix-valued Gegenbauer polynomials have the explicit expansion*

$$\begin{aligned} (P_n^{(\nu)}(x))_{k,i} &= \frac{(-2)^n}{i!} c_{k,0}^{(\nu)}(n) \sum_{j=i}^{2\ell} \frac{(n+k-j)! (-2\ell)_j (-1)^j (-n-k)_j}{(2\nu+2j)_{n+k-j} (2\nu+j+i-1)_{j-i} (2\nu+2\ell)_j} \\ &\times R_k(\lambda(j); -2\ell-1, -n-k-\nu, \nu-1, \nu-1) C_{n+k-j}^{(\nu+j)}(x) C_{j-i}^{(1-\nu-j)}(x). \end{aligned}$$

The right hand side in Corollary 6.3 is not obviously a polynomial of degree at most n in case $k > i$. In particular, the coefficients of x^p with $p > n$ are zero. In particular, for $k > i$ the leading coefficient of the right hand side is zero, and this gives

$$\begin{aligned} & \sum_{j=i}^{2\ell} \frac{(\nu+j)_{n+k-j} (-2\ell)_j (-1)^j (-n-k)_j}{(2\nu+2j)_{n+k-j} (2\nu+j+i-1)_{j-i} (2\nu+2\ell)_j} \\ & \times \frac{(1-\nu-j)_{j-i}}{(j-i)!} R_k(\lambda(j); -2\ell-1, -n-k-\nu, \nu-1, \nu-1) = 0. \end{aligned} \tag{6.7}$$

APPENDIX A. PROOF OF LDU -DECOMPOSITION

In Appendix A we prove Theorem 2.3. The proof is an extension of the result [15, Thm. 2.1, App. A] for the case $\nu = 1$. Note that the proof is verificational; it has initially been obtained by use of computer algebra for specific values of ν and ℓ . We indicate all the steps and we leave out the manipulation of shifted factorials. The limit case $\nu \rightarrow 0$ has to be excluded. So we restrict to $\nu > -\frac{1}{2}$, $\nu \neq 0$.

We start by recalling the orthogonality relations for the Gegenbauer (or ultraspherical) polynomials, see e.g. [1], [12], [13],

$$\begin{aligned} \int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} C_n^{(\nu)}(x) C_m^{(\nu)}(x) dx &= \delta_{n,m} \frac{(2\nu)_n \sqrt{\pi} \Gamma(\nu + \frac{1}{2})}{n! (n+\nu) \Gamma(\nu)}, \\ C_n^{(\nu)}(x) &= \frac{(2\nu)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, n+2\nu \\ \nu + \frac{1}{2} \end{matrix}; \frac{1-x}{2} \right). \end{aligned} \quad (\text{A.1})$$

By using the symmetry and taking matrix elements we see that the LDU-decomposition is equivalent to proving

$$\sum_{t=0}^m \alpha_t^{(\nu)}(m, n) C_{m+n-2t}^{(\nu)}(x) = \sum_{k=0}^m \frac{m! n! t_k^{(\nu)} (1-x^2)^k}{k! k! (2\nu+2k)_{m-k} (2\nu+2k)_{n-k}} C_{m-k}^{(\nu+k)}(x) C_{n-k}^{(\nu+k)} \quad (\text{A.2})$$

for $n \geq m$. Note first that the right hand side is a polynomial of degree $n+m$. We need to find its expansion in terms of Gegenbauer polynomials $C_r^{(\nu)}(x)$, so that we need the integral

$$\int_{-1}^1 C_{m-k}^{(\nu+k)}(x) C_{n-k}^{(\nu+k)}(x) C_r^{(\nu)}(x) (1-x^2)^{k+\nu-1/2} dx. \quad (\text{A.3})$$

Since the Gegenbauer polynomials are symmetric and by the orthogonality relations for the Gegenbauer polynomials $C_n^{(\nu+k)}$ we see that this integral is zero for $n+m-r$ odd and for $r < n-m$, so that the expansion of the right hand side of (A.2) in terms of Gegenbauer polynomials $C_r^{(\nu)}(x)$ only has non-zero terms for the summands in the left hand side of (A.2). So using (A.1) it remains to prove the identity

$$\begin{aligned} \alpha_t^{(\nu)}(m, n) \frac{(2\nu)_{m+n-2t} \sqrt{\pi} \Gamma(\nu + \frac{1}{2})}{(m+n-2t)! (m+n-2t+\nu) \Gamma(\nu)} = \\ \sum_{k=0}^m \frac{m! n! t_k^{(\nu)}}{k! k! (2\nu+2k)_{m-k} (2\nu+2k)_{n-k}} \int_{-1}^1 C_{m-k}^{(\nu+k)}(x) C_{n-k}^{(\nu+k)}(x) C_{m+n-2t}^{(\nu)}(x) (1-x^2)^{k+\nu-1/2} dx \end{aligned} \quad (\text{A.4})$$

for $n \geq m$, $t \in \{0, \dots, m\}$.

The integral in (A.4) has been evaluated in [15, Rmk. 2.8] and in order to prove (A.4) we follow the steps in [15, App. A]. The integral in (A.4) is equal to a balanced ${}_4F_3$ -series [15, Rmk. 2.8],

$$\begin{aligned} \frac{(\nu+k)_{m-k} (2\nu+2k)_{n-k} (-m)_{m-t} (\nu)_{n-t} \Gamma(\nu+k+1/2)}{(m-k)! (n-k)! (m-t)! \Gamma(\nu+m+n-t+1)} \sqrt{\pi} \\ \times {}_4F_3 \left(\begin{matrix} k-m, -m-k-2\nu+1, t-m, \nu+1+n-t \\ -m, -m-\nu+1, n-m+1 \end{matrix}; 1 \right) \end{aligned}$$

Using Whipple's transformation, see e.g. [1, Thm. 3.3], twice (once with (n, a, d) of [1, Thm. 3.3] as $(m-k, -m-2\nu-k+1, -m)$ and the second time as $(t, -m-n-\nu+t, -m)$)

the ${}_4F_3$ -series can be rewritten as

$$\frac{(1-n-m-2\nu)_{m-k}(-n)_t(\nu)_t}{(1+n-m)_{m-k}(1-\nu-m)_t(1-m-n-2\nu)_t} {}_4F_3 \left(\begin{matrix} -k, k+2\nu-1, -t, t-m-n-\nu \\ -m, -n, \nu \end{matrix}; 1 \right) \quad (\text{A.5})$$

We now plug the obtained expression for the integral into the right hand side of (A.4), where we replace the ${}_4F_3$ -series of (A.5) by its sum $\sum_{j=0}^k$. We next interchange the summations over k and j , and in the summation $\sum_{k=j}^m$ we replace $k = p + j$. This gives for the inner sum (the k -dependent part)

$$\begin{aligned} & \frac{(2\nu+2\ell)_j(1-n-m-2\nu)_{m-j}(2\nu+j)_j}{(-2\ell)_j(2\nu+j)_m(m-j)!(1+n-m)_{m-j}} \\ & \times {}_5F_4 \left(\begin{matrix} 2\nu+2j-1, \nu+j+\frac{1}{2}, j-n, 2\nu+2\ell+j, j-m \\ \nu+j-\frac{1}{2}, 2\nu+j+n, -2\ell+j, 2\nu+m+j \end{matrix}; 1 \right) \\ & = \frac{(1-n-m-2\nu)_m(n-2\ell)_m}{(-2\ell)_m m! (1+n-m)_m (2\nu+n)_m} \frac{(2\nu+2\ell)_j(-m)_j(-n)_j}{(1-n-m+2\ell)_j} \end{aligned}$$

using the Dougall summation formula for a very-well-poised ${}_5F_4$ -series, see e.g. [1, Cor. 4.3], [2, §4.3(3)]. This shows that the right hand side of (A.4) can be written as a single sum; explicitly

$$\begin{aligned} & \frac{(-m)_{m-t}(\nu)_{n-t}\sqrt{\pi}(-n)_t(\nu)_t n! (2\ell+\nu)(\nu)_m \Gamma(\nu+\frac{1}{2})(1-n-m-2\nu)_m (n-2\ell)_m}{(n-m)!(m-t)!\Gamma(\nu+n+m-t+1)(1-\nu-m)_t(1-n-m-2\nu)_t(-2\ell)_m} \\ & \times \frac{1}{(1+n-m)_m(2\nu+n)_m} {}_3F_2 \left(\begin{matrix} -t, t-m-n-\nu, 2\nu+2\ell \\ \nu, 1-n-m+2\ell \end{matrix}; 1 \right) \end{aligned}$$

The balanced ${}_3F_2$ -series is summable to $\frac{(1-m-n-2\nu)_t(-2\ell-\nu)_t}{(\nu)_t(2\ell+1-m-n)_t}$ by the Pfaff-Saalschütz formula, see e.g. [1, Thm. 2.2.6], [2, §2.2], [12, (1.4.5)]. Next a straightforward verification using the expression of Definition 2.1 shows that this is equal to the left hand side of (A.4), which proves Theorem 2.3.

The proof remains valid for $-\frac{1}{2} < \nu < 0$.

APPENDIX B. PROOF OF THE STATEMENTS IN SECTION 3

Recall that for $\nu = 1$ we have two different proofs of the symmetry of the matrix-valued differential operators of Theorem 3.1, one by direct calculation [14, App. 2] and one from the group theoretic interpretation [15, §7]. In this case we prove Theorem 3.1 along the lines of [14, App. 2], since we do not have a group theoretic interpretation available.

Proof of Theorem 3.1. In order to verify (3.1) and (3.2), we proceed as in [14, Thm. 7.5]. In terms of $\rho(x) = (1 - x^2)^{\nu-1/2}$ and $W_{\text{pol}}^{(\nu)}(x)$ as in Definition 2.1, (3.1) and (3.2) are given by

$$0 = W_{\text{pol}}^{(\nu)}(x)(xB_1^{(\nu)} + B_0^{(\nu)})^* + (xB_1^{(\nu)} + B_0^{(\nu)})W_{\text{pol}}^{(\nu)}(x), \quad (\text{B.1})$$

$$\begin{aligned} 0 = & -A_1'(x)W_{\text{pol}}^{(\nu)}(x) - \frac{\rho'(x)}{\rho(x)}(xB_1^{(\nu)} + B_0^{(\nu)})W_{\text{pol}}^{(\nu)}(x) \\ & - (xB_1^{(\nu)} + B_0^{(\nu)})W_{\text{pol}}^{(\nu)}(x)' + A_0W_{\text{pol}}^{(\nu)}(x) - W_{\text{pol}}^{(\nu)}(x)A_0^{(\nu)}. \end{aligned} \quad (\text{B.2})$$

We verify the conditions above for all the (m, n) -entries with $n < m$. The other cases can be done similarly. Now taking matrix elements, we see that (B.1) is a polynomial identity, which we expand into the basis of Gegenbauer polynomials $C_n^{(\nu)}(x)$. Then the (m, n) -entry of (B.1) holds true if and only if

$$\begin{aligned} 0 = & -\frac{n}{2\ell} \sum_{t=0}^{n-1} \alpha_t^{(\nu)}(m, n-1) C_{n+m-2t-1}^{(\nu)}(x) - \frac{\ell-n}{\ell} \sum_{t=0}^n x \alpha_t^{(\nu)}(m, n) C_{n+m-2t}^{(\nu)}(x) \\ & + \frac{2\ell-n}{2\ell} \sum_{t=0}^{n+1} \alpha_t^{(\nu)}(m, n+1) C_{n+m-2t+1}^{(\nu)}(x) - \frac{m}{2\ell} \sum_{t=0}^n \alpha_t^{(\nu)}(m-1, n) C_{n+m-2t-1}^{(\nu)}(x) \\ & - \frac{\ell-m}{\ell} \sum_{t=0}^n x \alpha_t^{(\nu)}(m, n) C_{n+m-2t}^{(\nu)}(x) + \frac{2\ell-m}{2\ell} \sum_{t=0}^n \alpha_t^{(\nu)}(m+1, n) C_{n+m-2t+1}^{(\nu)}(x) \end{aligned}$$

where we have used the explicit expressions from Definition 2.1 and that $xB_1^{(\nu)} + B_0^{(\nu)}$ is tridiagonal. Next we use the three-recurrence relation for Gegenbauer polynomials, see e.g. [12], [13],

$$xC_r^{(\nu)}(x) = \frac{(r+1)}{2(r+\nu)} C_{r+1}^{(\nu)}(x) + \frac{(r+2\nu-1)}{2(r+\nu)} C_{r-1}^{(\nu)}$$

to get rid of the multiplication by x . It is then immediate from the expression of the coefficient $\alpha_t^{(\nu)}(m, n)$ that the coefficient of $C_{m-n-1}^{(\nu)}$ vanishes. Regrouping and using the explicit

expressions for $\alpha_t^{(\nu)}(m, n)$ we get that the right hand side of the previous formula equals

$$\begin{aligned} \sum_{t=0}^m \alpha_t^{(\nu)}(m, n) \frac{(m+n-2t+2\nu-1)}{2\ell(m+n-2t+\nu)} C_{n+m-2t-1}^{(\nu)}(x) \\ \times \left[\frac{(m+n-2t+\nu-1)(n+m+\nu-t)(2\ell-n+1)(n-t)}{(2\ell-n-m+t+1)(m+n-2t)(n+\nu-t-1)} \right. \\ + \frac{(m+n-2t+\nu-1)(n+m+\nu-t)(2\ell-m+1)(m-t)}{2\ell(2\ell-n-m+t+1)(m+n-2t)(m+\nu-t-1)} \\ - (2\ell-n-m) + \frac{(m+n-1-2t+\nu)(2\ell+\nu-t)(n+1)(m-t)}{(m+n-2t)(t+1)(m+\nu-t-1)} \\ + \frac{(m+n-2t+\nu-1)(2\ell+\nu-t)(m+1)(n-t)}{(m+n-2t)(t+1)(n+\nu-t-1)} \\ \left. - \frac{(2\ell-m-n)(m+n-2t+2\nu-2)(n-t)(n+m+\nu-t)(m-t)(2\ell+\nu-t)}{(m+\nu-t-1)(m+n-2t)(\nu+n-t-1)(2\ell-m-n+t+1)(t+1)} \right], \end{aligned}$$

and it is a straightforward check that the coefficient in square brackets is equal to 0.

The (m, n) -entry of (B.2) is given by

$$\begin{aligned} (2\nu-1)x((xB_1^{(\nu)} + B_0^{(\nu)})W_{\text{pol}}^{(\nu)}(x))_{m,n} - (1-x^2)((xB_1^{(\nu)} + B_0^{(\nu)})W_{\text{pol}}^{(\nu)}(x'))_{m,n} \\ + (1-x^2)[(A_0^{(\nu)})_{m,m} - (A_0^{(\nu)})_{n,n} - (B_1^{(\nu)})_{m,m}](W_{\text{pol}}^{(\nu)}(x))_{m,n} = 0 \end{aligned}$$

Again we plug in Definition 2.1, and we expand all sums in Gegenbauer polynomials $C_k^{(\nu)}(x)$. So we use the three-term recurrence for the Gegenbauer polynomials to get rid of the multiplication by x and by x^2 . We also have to use

$$(1-x^2)\frac{dC_n^{(\nu)}}{dx}(x) = \frac{(n+2\nu-1)(n+2\nu)}{2(n+\nu)}C_{n-1}^{(\nu)}(x) - \frac{n(n+1)}{2(n+\nu)}C_{n+1}^{(\nu)}(x) \quad (\text{B.3})$$

see e.g. [12, (4.5.7)] with the convention $C_{-1}^{(\nu)}(x) = 0$. So this gives again a sum only involving Gegenbauer polynomials as in the check of the (B.1). Insert the explicit expression of the coefficients $\alpha_t^{(\nu)}(m, n)$ to complete the proof by a straightforward but involved computation to see that each coefficient in the expansion of Gegenbauer polynomials vanishes.

Now we show that the differential operator $D^{(\nu)}$ is symmetric, so that we need to check (3.4) and (3.5). The computation to check (3.4) is similar to that of (B.2) for the operator $E^{(\nu)}$.

Now we give the proof for (3.5). We prove instead that

$$\begin{aligned} W^{(\nu)}(x)(C^{(\nu)} - xU^{(\nu)})^* - (C^{(\nu)} - xU^{(\nu)})W^{(\nu)}(x) \\ + 2\left(\int W^{(\nu)}(x)dx\right)V^{(\nu)} - 2V^{(\nu)}\left(\int W^{(\nu)}(x)dx\right) = 0, \end{aligned} \quad (\text{B.4})$$

which is obtained by integrating (3.5) with respect to x . Then (3.5) follows by taking the derivative with respect to x .

We assume $m < n$. The other cases can be proved similarly. Let $\rho(x) = (1 - x^2)^{\nu-1/2}$ be the scalar weight. We have

$$\begin{aligned}
& (W(x)(C^{(\nu)} - xU^{(\nu)})^* - (C^{(\nu)} - xU^{(\nu)})W(x))_{m,n} \\
&= \rho(x) \sum_{t=0}^m C_{m+n-2t-1}^{(\nu)} \left[n\alpha_t^{(\nu)}(m, n-1) + (2\ell - n)\alpha_{t+1}^{(\nu)}(m, n+1) \right. \\
&\quad \left. - m\alpha_{t+1}^{(\nu)}(m-1, n) - (2\ell - m)\alpha_{t+1}^{(\nu)}(m+1, n) \right] \\
&= - \sum_{t=0}^m \rho(x) C_{m+n-2t-1}^{(\nu)} [(\nu-1)(mn + m - 2t\ell - 2t - 2\ell - 1 + n) + 2(n-t)(m-t)(\ell+1)] \\
&\quad \times \frac{(2\ell - n - m)(m - n)(m + n - 1 - 2t + \nu)(m + n + 2\nu - 1 - 2t)}{(m + n - 2t + \nu)(m + n - 2t)(m + \nu - t - 1)(2\ell - n - m + t + 1)(n + \nu - t - 1)}. \quad (\text{B.5})
\end{aligned}$$

In order to compute the part of (B.4) involving the integrals, we use the following formula for the Gegenbauer polynomials;

$$\int \rho(x) C_r^{(\nu)}(x) dx = \rho(x) \left(\frac{(r+1)C_{r+1}^{(\nu)}(x)}{2(\nu+r)(2\nu+r)} - \frac{(2\nu+r-1)C_{r-1}^{(\nu)}(x)}{2r(\nu+r)} \right), \quad (\text{B.6})$$

which follows using the Rodrigues formula for the Gegenbauer polynomials [12, (4.5.11)], next [12, (4.5.5)] and (B.3). Using (B.6) for the integrals in (B.4) we obtain

$$\begin{aligned}
& 2 \left(\left(\int W^{(\nu)}(x) dx \right) V^{(\nu)} - V^{(\nu)} \left(\int W^{(\nu)}(x) dx \right) \right)_{m,n} \\
&= (m-n)(2\ell - m - n)\rho(x) \sum_{t=0}^m \alpha_t^{(\nu)}(m, n) \left[-\frac{(2\nu + m + n - 2t - 1)}{(m + n - 2t)(\nu + m + n - 2t)} \right. \\
&\quad \left. \times \frac{\alpha_t^{(\nu)}(m, n)}{\alpha_{t+1}^{(\nu)}(m, n)} \frac{(m + n - 2t - 1)}{(\nu + m + n - 2t - 2)(2\nu + m + n - 2t - 2)} \right] C_{m+n-2t-1}^{(\nu)}(x),
\end{aligned}$$

which is exactly the negative of (B.5). This completes the proof of Theorem 3.1. \square

Proof of Theorem 3.4. The proof of Theorem 3.4 uses the same ingredients as the proof of Theorem 3.1 in Appendix B. Let us calculate the (m, n) entry of both sides in (3.10). We assume $m < n$, the other situations can be treated analogously. Now $(W^{(\nu)}(x)\Psi^{(\nu)}(x))_{m,n}$ can be written explicitly using Definition 2.1 and (3.8). This gives explicit terms involving Gegenbauer polynomials multiplied by x , and using the three-term recurrence relation for Gegenbauer polynomials and regrouping shows that we can write $(W^{(\nu)}(x)\Psi^{(\nu)}(x))_{m,n}$ as an expansion in terms of Gegenbauer polynomials;

$$\frac{(2\nu+1)(2\ell+\nu+1)}{\nu(2\ell+\nu)(\ell+\nu)} \sum_{t=0}^m \eta_t^{(\nu)}(m, n) C_{m+n-2t+1}^{(\nu)}(x) \quad (\text{B.7})$$

where

$$\begin{aligned}\eta_t^{(\nu)}(n, m) &= \frac{-(\nu+n)(\nu+2\ell-n)(m+n-2t+1)}{n+m-2t+\nu} \alpha_t^{(\nu)}(m, n) \\ &\quad - (n-2\ell)(\nu+n) \alpha_t^{(\nu)}(m, n+1) + n(\nu+2\ell-n) \alpha_t^{(\nu)}(m, n-1) \\ &\quad - \frac{(\nu+n)(\nu+2\ell-n)(2\nu+m+n-2t+1)}{n+m-2t+\nu+2} \alpha_{t-1}^{(\nu)}(m, n), \\ \eta_0^{(\nu)}(m, n) &= \frac{-(\nu+n)(\nu+2\ell-n)(n+m+1)}{(n+m+\nu)} \alpha_0^{(\nu)}(m, n) - (n-2\ell)(\nu+n) \alpha_0^{(\nu)}(m, n+1), \\ \eta_{m+1}^{(\nu)}(m, n) &= n(\nu+2\ell-n) \alpha_m^{(\nu)}(m, n-1) - \frac{(\nu+n)(\nu+2\ell-n)}{(n-m+\nu)} \alpha_m^{(\nu)}(m, n)\end{aligned}$$

for $t \in \{1, \dots, m\}$. On the other hand, by (B.3) we can expand $(W^{(\nu+1)}(x))'_{m,n}$ in the same basis leading to

$$(1-x^2)^{\nu-1/2} \sum_{t=0}^m \frac{-(n+m-2t+1)(2\nu+n+m-2t+1)}{2\nu} \alpha_t^{(\nu+1)}(m, n) C_{m+n-2t+1}^{(\nu)}(x). \quad (\text{B.8})$$

This can be done by calculating

$$\int_{-1}^1 C_k^{(\nu)}(x) (W_{m,n}^{(\nu+1)}(x))' dx = - \int_{-1}^1 \frac{dC_k^{(\nu)}}{dx}(x) (1-x^2)^{\nu+1/2} \sum_{t=0}^m \alpha_t^{(\nu+1)}(m, n) C_{m+n-2t}^{(\nu+1)} dx$$

and since $\frac{dC_k^{(\nu)}}{dx}(x) = 2\nu C_{k-1}^{(\nu+1)}$ by [12, (4.5.5)], the integral can be calculated using (A.1), leading to (B.8).

By a straightforward computation we check that (B.7) and (B.8) are the same up to the constant as given in Theorem 3.4. \square

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